

Algebra

CHAPTER 3

Complex Numbers

3.1–3.52

Introduction	3.1
Imaginary Number	3.1
Integral Powers of Iota (i)	3.1
<i>Concept Application Exercise 3.1</i>	3.2
Complex Number	3.2
Representation of Complex Numbers	3.2
Algebraic Operations on Complex Numbers	3.3
Conjugate of Complex Number	3.4
Section Formula in Complex Numbers	3.5
<i>Concept Application Exercise 3.2</i>	3.6
Equality of Complex Numbers	3.6
Square Root of Complex Number	3.7
<i>Concept Application Exercise 3.3</i>	3.9
Cube Roots of Unity	3.9
Properties of Cube Roots of Unity	3.9
<i>Concept Application Exercise 3.4</i>	3.11
Polar Form of Complex Number	3.11
Modulus of Complex Number	3.11
Argument of Complex Number	3.11
Euler's Form of Complex Number	3.15
Logarithm of Complex Number	3.15
<i>Concept Application Exercise 3.5</i>	3.16
Multiplication and Division of Complex Numbers in Polar Form	3.16
<i>Concept Application Exercise 3.6</i>	3.18

De Moivre's Theorem	3.19
<i>Concept Application Exercise 3.7</i>	3.20
Important Properties of Modulus	3.20
<i>Concept Application Exercise 3.8</i>	3.24
Locus Based on Distance Formula	3.24
<i>Concept Application Exercise 3.9</i>	3.27
Angle between Two Lines and Rotation Formula	3.27
Angle between Two Rays	3.27
Equation of Ray	3.27
Equation of Straight Line	3.30
Condition for Three Points to Represent Equilateral Triangle	3.31
Equation of Arc	3.32
Equation of Circle whose Diametric End Points are Given	3.32
Condition for Four Points to be Concyclic	3.32
Rotation Formula	3.33
<i>Concept Application Exercise 3.10</i>	3.34
The n th Roots of Unity	3.35
Properties of n th Roots of Unity	3.35
<i>Concept Application Exercise 3.11</i>	3.37
<i>Exercises</i>	3.38
<i>Single Correct Answer Type</i>	3.38
<i>Multiple Correct Answers Type</i>	3.42
<i>Linked Comprehension Type</i>	3.45
<i>Matrix Match Type</i>	3.47
<i>Numerical Value Type</i>	3.48
<i>Archives</i>	3.49
<i>Answers Key</i>	3.52

3

Complex Numbers

INTRODUCTION

We know that equation $x^2 + 1 = 0$ or $x^2 = -1$ has no solution in real numbers, as there is no real number whose square is negative. So, we need to extend the real number system to a larger system so that we can find the solution of such equations. In this chapter, we will define the system of complex numbers which shall accommodate not only real numbers but also square root of negative numbers.

IMAGINARY NUMBER

Consider the equation $x^2 + 1 = 0$.

$$\therefore x^2 = -1$$

$$\therefore x = \pm \sqrt{-1}. \text{ Here } \sqrt{-1} \text{ is not a real number.}$$

So, we call it an imaginary number and denote it by Greek letter iota, i .

Thus, we have $i = \sqrt{-1}$, where $i^2 = -1$.

Here ' i ' is an imaginary number whose square is ' -1 '.

Now, any square root of negative number can be expressed in terms of i .

$$\text{e.g., } \sqrt{-4} = \sqrt{4} \sqrt{-1} = 2i, \sqrt{-25} = \sqrt{25} \sqrt{-1} = 5i, \\ \sqrt{-7} = \sqrt{7} \sqrt{-1} = 7i, \text{ etc.}$$

It should be noted that $\sqrt{ab} = \sqrt{a} \sqrt{b}$ only if at least one of the values a and b is non-negative.

If both a and b are negative, then using this property, we get some absurd results.

See the following calculations.

$$6 = \sqrt{36} = \sqrt{(-4) \times (-9)} \\ = \sqrt{-4} \sqrt{-9} \quad (\text{We can't use } \sqrt{ab} = \sqrt{a} \sqrt{b}, \\ \text{when both } a \text{ and } b \text{ are negative}) \\ = \sqrt{4} \sqrt{-1} \sqrt{9} \sqrt{-1} \\ = 2i \times 3i = 6i^2 = -6$$

Because of using wrong mathematical concept in one step we are getting $6 = -6$.

INTEGRAL POWERS OF IOTA (i)

Since $i = \sqrt{-1}$, we have $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$.

To find the value of i^n ($n > 4$), first divide n by 4.

Let q be the quotient and r be the remainder.

$$\text{i.e., } n = 4q + r, \text{ where } 0 \leq r \leq 3$$

$$i^n = i^{4q+r} = (i^4)^q (i)^r = (1)^q (i)^r = i^r$$

In general, we have $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, where n is any integer.

Also, we have $2i + 3i = 5i$, $6i - 2i = 4i$ etc.

ILLUSTRATION 3.1

Evaluate:

$$(i) \ i^{135}$$

$$(ii) \ \frac{1}{i^{47}}$$

$$(iii) \ (-\sqrt{-1})^{4n+3}, n \in N \quad (iv) \ \sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9}$$

Sol.

$$(i) \ 135 \text{ leaves remainder as 3 when it is divided by 4. Therefore,} \\ i^{135} = i^3 = -i$$

$$(ii) \ \frac{1}{i^{47}} = \frac{1}{i^{44} i^3} = \frac{1}{-i} = \frac{i}{-i^2} = i$$

$$(iii) \ (-\sqrt{-1})^{4n+3} = (-i)^{4n+3} \\ = (-i)^{4n} (-i)^3 \\ = \{(-i)^4\}^n (-i)^3 \\ = 1 \times (-i)^3 = -i^3 = -(-i) = i$$

$$(iv) \ \sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9} = 5i + 6i + 6i = 17i$$

ILLUSTRATION 3.2

Find the value of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ for all $n \in N$.

$$\text{Sol. } i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n [1 + i + i^2 + i^3] \\ = i^n [1 + i - 1 - i] \\ = i^n (0) = 0$$

ILLUSTRATION 3.3

Find the value of $1 + i^2 + i^4 + i^6 + \dots + i^{2n}$, $n \in N$.

$$\text{Sol. } S = 1 + i^2 + i^4 + \dots + i^{2n} = 1 - 1 + 1 - 1 + \dots + (-1)^n$$

Obviously it depends on n . Hence, it cannot be determined unless n is known.

ILLUSTRATION 3.4

Show that the polynomial $x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$ is divisible by $x^3 + x^2 + x + 1$, where $p, q, r, s \in N$.

$$\text{Sol. } \text{Let } f(x) = x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}. \text{ Now,}$$

$$x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1)$$

$$= (x^2 - i^2)(x + 1)$$

$$= (x + i)(x - i)(x + 1)$$

$$\text{Now, } f(i) = i^{4p} + i^{4q+1} + i^{4r+2} + i^{4s+3} \\ = 1 + i^1 + i^2 + i^3 \\ = 1 + i - 1 - i = 0$$

$$\begin{aligned}
 f(-i) &= (-i)^{4p} + (-i)^{4q+1} + (-i)^{4r+2} + (-i)^{4s+3} \\
 &= 1 + (-i)^1 + (-i)^2 + (-i)^3 \\
 &= 1 - i - 1 + i = 0 \\
 f(-1) &= (-1)^{4p} + (-1)^{4q+1} + (-1)^{4r+2} + (-1)^{4s+3} = 0
 \end{aligned}$$

Thus, by factor theorem $f(x)$ is divisible by $x^3 + x^2 + x + 1$.

ILLUSTRATION 3.5

Solve $ix^2 - 3x - 2i = 0$.

Sol. $ix^2 - 3x - 2i = 0$,
or $x^2 + 3ix - 2 = 0$ (dividing by i)
or $x = \frac{-3i \pm \sqrt{-9 + 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{-3i \pm \sqrt{-1}}{2} = \frac{-3i \pm i}{2}$
 $\Rightarrow x = -i$ or $x = -2i$

ILLUSTRATION 3.6

If $z = 4 + i\sqrt{7}$, then find the value of $z^3 - 4z^2 - 9z + 91$.

Sol. $z = 4 + i\sqrt{7}$
or $z - 4 = i\sqrt{7}$
or $z^2 - 8z + 16 = -7$ (Squaring)
or $z^2 - 8z + 23 = 0$
Now dividing $z^3 - 4z^2 - 9z + 91$ by $z^2 - 8z + 23$, we get
 $z^3 - 4z^2 - 9z + 91 = (z^2 - 8z + 23)(z + 4) - 1 = -1$

CONCEPT APPLICATION EXERCISE 3.1

- Is the following computation correct? If not give the correct computation:
 $\sqrt{(-2)} \sqrt{(-3)} = \sqrt{(-2)(-3)} = \sqrt{6}$
- Find the value of
 $\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} - 1$.
- The value of $i^{1+3+5+\dots+(2n+1)}$ is _____.
- Find the value of $x^4 + 9x^3 + 35x^2 - x + 4$ for $x = -5 + 2\sqrt{-4}$.

ANSWERS

1. Not Correct 2. -2 3. $z = \begin{cases} 1, & \text{if } n \text{ is odd} \\ i, & \text{if } n \text{ is even} \end{cases}$ 4. -160

COMPLEX NUMBER

Consider quadratic equation $x^2 - 2x + 10 = 0$.
Solving it, we get

$$\begin{aligned}
 x &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(10)}}{2} \\
 &= \frac{2 \pm \sqrt{-36}}{2} \\
 &= \frac{2 \pm \sqrt{36}\sqrt{-1}}{2} \\
 &= \frac{2 \pm 6i}{2}
 \end{aligned}$$

$\therefore x = 1 + 3i, 1 - 3i$

Here, we get the number $1 + 3i$ as a root of quadratic equation. This number is combination of two numbers, real number '1' and imaginary number '3i'. Here, we have '+' or '-' sign between real and imaginary numbers. But actually, we can't add or subtract real number and imaginary number. These signs are considered as symbols which connect two different types of numbers, which have some geometrical interpretations. Later, we will be devising a scheme for geometrical interpretation of complex numbers.

In standard form, complex number z is expressed as $a + ib$ where $a, b \in R$.

Here, 'a' is real part or real number, 'b' is imaginary part and number 'ib' is imaginary number. Don't confuse between imaginary part and imaginary number.

Complex number $a + ib$ can also be thought of as ordered pair (a, b) .

Real part is denoted by, $a = \text{Re}(z)$ and imaginary part is denoted by, $b = \text{Im}(z)$.

Now, we define a set of complex numbers C , which contains such complex numbers as

$$C = \{a + ib; a, b \in R, i = \sqrt{-1}\}.$$

Some examples of complex numbers are $5 + 3i, -1 + i, 0 + 4i, 4 + 0i$, etc..

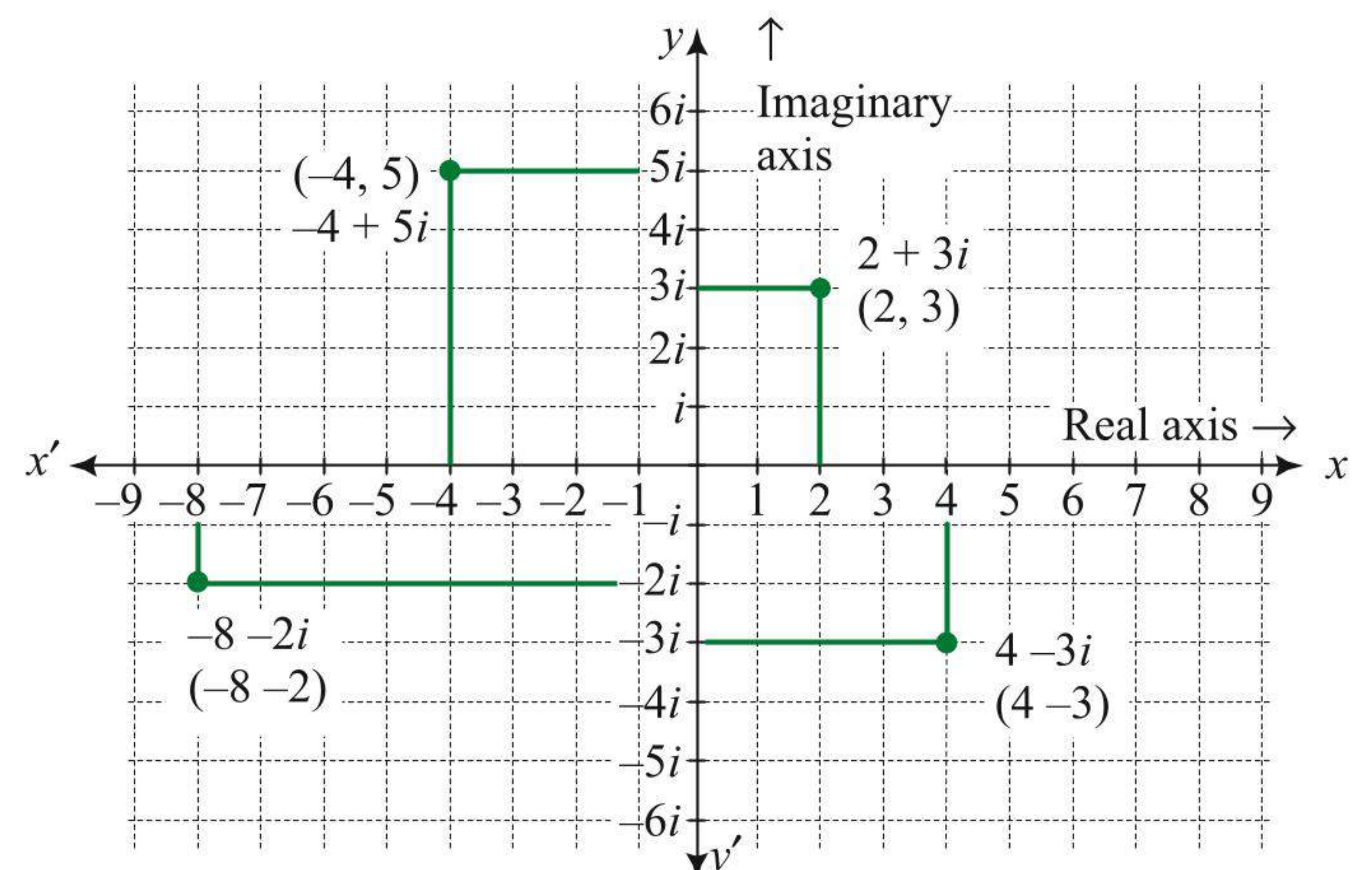
In $z = a + ib$, if $b = \text{Im}(z) = 0$, then $z = a$, which is called purely real number.

If $a = \text{Re}(z) = 0$, then $z = ib$, which is called purely imaginary number.

Thus, set of complex numbers C contains all the real numbers. So, set of real numbers R is subset of set C , i.e., $R \subset C$.

REPRESENTATION OF COMPLEX NUMBERS

Consider a complex number $z = a + ib$. There may be several ways to write the combination of two numbers a and ib , one of which is ordered pair (a, b) , where first element is real part of z and second element is imaginary part of z . Since these ordered pairs are represented as points on the coordinate plane, it leads to the idea for representation of complex numbers. For complex numbers, the coordinate plane changes to the complex plane or **Argand plane** where y -axis is named as 'imaginary axis' and x -axis is named as 'real axis'. Complex number ' $a + ib$ ' is represented as a point (a, b) on this plane. In the following figure, several complex numbers in different quadrants are represented.



On real axis, we have purely real numbers and on imaginary axis, we have purely imaginary numbers.

Note:

- Multiplication of a non-zero complex number by i rotates the point about origin through a right angle in the anticlockwise direction.

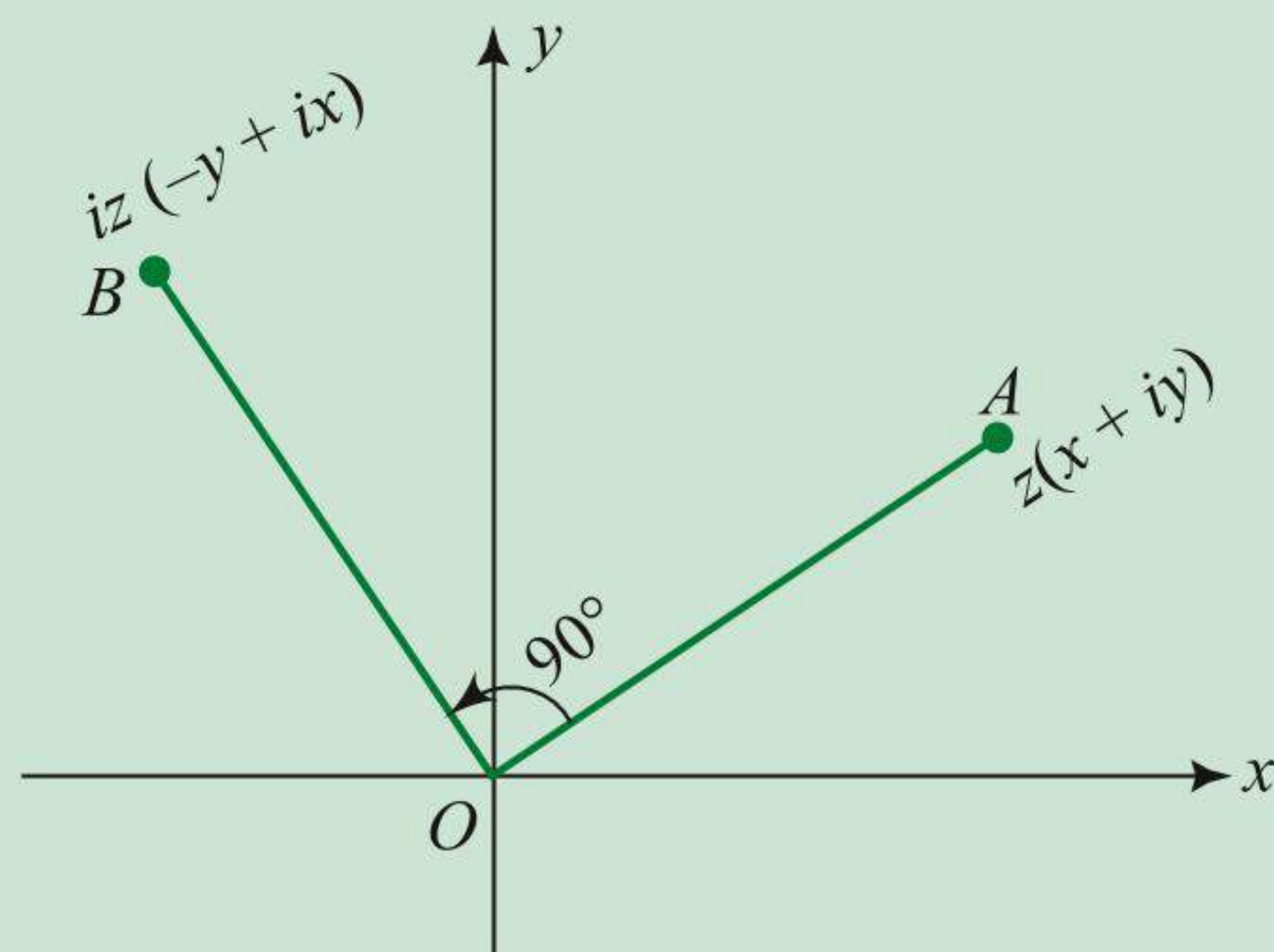
Proof: Let $z = x + iy$, where $x, y > 0$

i.e., z or point $A(x, y)$ lies in first quadrant.

Now, $iz = i(x + iy) = ix + i^2y = -y + ix$

Point $B(-y, x)$ lies in fourth quadrant.

Also, $\angle AOB = 90^\circ$.



Thus, B is obtained by rotating A in anticlockwise direction about origin.

- Multiplication of a non-zero complex number by $-i$ rotates the point about origin through a right angle in the clockwise direction.

ALGEBRAIC OPERATIONS ON COMPLEX NUMBERS

Let two complex numbers be $z_1 = a + ib$ and $z_2 = c + id$.

Addition:

$$(z_1 + z_2) = (a + ib) + (c + id) = (a + c) + i(b + d)$$

Subtraction:

$$(z_1 - z_2) = (a + ib) - (c + id) = (a - c) + i(b - d)$$

Multiplication:

$$(z_1 \times z_2) = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

If $z = a + ib$, then

$$z^2 = (a + ib)^2 = a^2 + i^2b^2 + 2iab$$

$$z^3 = (a + ib)^3 = a^3 + i^3b^3 + 3abi(a + ib) \text{ etc.}$$

Division:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a + ib}{c + id} \quad (\text{where at least one of } c \text{ and } d \text{ is non-zero}) \\ &= \frac{(a + ib)}{(c + id)} \cdot \frac{(c - id)}{(c - id)} = \frac{(ac + bd)}{c^2 + d^2} + \frac{i(bc - ad)}{c^2 + d^2} \end{aligned}$$

ILLUSTRATION 3.7

Express each one of the following in the standard form $a + ib$.

$$(i) \frac{5+4i}{4+5i} \quad (ii) \frac{(1+i)^2}{3-i} \quad (iii) \frac{1}{1-\cos\theta + 2i\sin\theta}$$

Sol.

$$\begin{aligned} (i) \frac{5+4i}{4+5i} &= \frac{5+4i}{4+5i} \times \frac{4-5i}{4-5i} \\ &= \frac{(20+20) + i(16-25)}{16-25i^2} = \frac{40-9i}{41} = \frac{40}{41} - \frac{9}{41}i \end{aligned}$$

$$\begin{aligned} (ii) \frac{(1+i)^2}{3-i} &= \frac{1+2i+i^2}{3-i} \\ &= \frac{2i}{3-i} = \frac{2i}{3-i} \times \frac{3+i}{3+i} = \frac{6i+2i^2}{9-i^2} \\ &= \frac{-2+6i}{10} = -\frac{1}{5} + \frac{3}{5}i \end{aligned}$$

$$\begin{aligned} (iii) \frac{1}{1-\cos\theta + 2i\sin\theta} &= \frac{1}{1-\cos\theta + 2i\sin\theta} \times \frac{1-\cos\theta - 2i\sin\theta}{1-\cos\theta - 2i\sin\theta} \\ &= \frac{1-\cos\theta - 2i\sin\theta}{(1-\cos\theta)^2 + 4\sin^2\theta} \\ &= \frac{1-\cos\theta - 2i\sin\theta}{1-2\cos\theta + \cos^2\theta + 4\sin^2\theta} \\ &= \frac{1-\cos\theta - 2i\sin\theta}{2-2\cos\theta + 3\sin^2\theta} \\ &= \left(\frac{1-\cos\theta}{2-2\cos\theta + 3\sin^2\theta} \right) + i \left(\frac{-2\sin\theta}{2-2\cos\theta + 3\sin^2\theta} \right) \end{aligned}$$

ILLUSTRATION 3.8

Solve $2(1+i)x^2 - 4(2-i)x - 5 - 3i = 0$.

Sol. $2(1+i)x^2 - 4(2-i)x - 5 - 3i = 0$

$$\begin{aligned} \Rightarrow x &= \frac{4(2-i) \pm \sqrt{16(2-i)^2 + 4 \cdot 2 \cdot (1+i) \cdot (5+3i)}}{4(1+i)} \\ &= \frac{4(2-i) \pm \sqrt{16(4-4i-1) + 8(5+3i+5i-3)}}{4(1+i)} \\ &= \frac{4(2-i) \pm \sqrt{48-64i+16+64i}}{4(1+i)} \\ &= \frac{4(2-i) \pm 8}{4(1+i)} \\ &= \frac{(2-i) \pm 2}{1+i} \\ &= \frac{2-i+2}{1+i}; \frac{2-i-2}{1+i} \\ &= \frac{(4-i)(1-i)}{2}; \frac{-i(1-i)}{2} \\ &= \frac{3-5i}{2}; -\left(\frac{1+i}{2}\right) \end{aligned}$$

ILLUSTRATION 3.9

Find the value of $(1+i)^6 + (1-i)^6$.

$$\begin{aligned} \text{Sol. } (1+i)^6 + (1-i)^6 &= [(1+i)^2]^3 + [(1-i)^2]^3 \\ &= [1+2i+i^2]^3 + [1-2i+i^2]^3 \\ &= (2i)^3 + (-2i)^3 \\ &= (8-8)i^3 = 0 \end{aligned}$$

ILLUSTRATION 3.10

If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of m .

Sol. $\left(\frac{1+i}{1-i}\right)^m = 1$

or $\left(\frac{(1+i)^2}{1-i^2}\right)^m = 1$

or $\left(\frac{1+i^2+2i}{2}\right)^m = 1$

or $i^m = 1$

Therefore, the least positive integral value of m is 4.

ILLUSTRATION 3.11

Prove that the triangle formed by the points $1, \frac{1+i}{\sqrt{2}}$ and i as vertices in the Argand diagram is isosceles.

Sol. The vertices of the triangle are $A(1)$, $B\left(\frac{1+i}{\sqrt{2}}\right)$ and $C(i)$.

i.e., vertices of the triangle are $A(1, 0)$, $B\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $C(0, 1)$.

$$\therefore AB^2 = \left(1 - \frac{1}{\sqrt{2}}\right)^2 + \left(0 - \frac{1}{\sqrt{2}}\right)^2 = 2 - \sqrt{2}$$

$$BC^2 = \left(\frac{1}{\sqrt{2}} - 0\right)^2 + \left(\frac{1}{\sqrt{2}} - 1\right)^2 = 2 - \sqrt{2}$$

Thus, $AB = BC$.

So, triangle is isosceles

ILLUSTRATION 3.12

Find the values of θ if $(3 + 2i \sin \theta)/(1 - 2i \sin \theta)$ is purely real or purely imaginary.

Sol. $z = \frac{3 + 2i \sin \theta}{1 - 2i \sin \theta} = \frac{3 + 2i \sin \theta}{1 - 2i \sin \theta} \cdot \frac{1 + 2i \sin \theta}{1 + 2i \sin \theta}$

$$z = \frac{(3 + 2i \sin \theta)(1 + 2i \sin \theta)}{1 + 4 \sin^2 \theta}$$

$$= \frac{3 - 4 \sin^2 \theta + 8i \sin \theta}{1 + 4 \sin^2 \theta}$$

Now z is purely real if $\sin \theta = 0$ or $\theta = n\pi, n \in \mathbb{Z}$.

Also, z is purely imaginary if

$$3 - 4 \sin^2 \theta = 0$$

or $\sin \theta = \pm \frac{\sqrt{3}}{2} = \pm \sin \frac{\pi}{3}$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$$

ILLUSTRATION 3.13

If the imaginary part of $(2z + 1)/(iz + 1)$ is -2 , then find the locus of the point z representing in the complex plane.

Sol. Let $z = x + iy$

$$\begin{aligned} \Rightarrow \frac{2z + 1}{iz + 1} &= \frac{2(x + iy) + 1}{i(x + iy) + 1} \\ &= \frac{(2x + 1) + i2y}{1 - y + ix} \\ &= \frac{(2x + 1) + i2y}{(1 - y) + ix} \cdot \frac{(1 - y) - ix}{(1 - y) - ix} \\ &= \frac{(2x + 1)(1 - y) + 2xy + i[-x(2x + 1) + 2y(1 - y)]}{(1 - y)^2 + x^2} \end{aligned}$$

Since imaginary part of $(2z + 1)/(iz + 1)$ is -2 , we have

$$\frac{-x(2x + 1) + 2y(1 - y)}{(1 - y)^2 + x^2} = -2$$

or $-2x^2 - x + 2y - 2y^2 = -2[1 + y^2 - 2y + x^2]$

or $x + 2y - 2 = 0$, which is a straight line

CONJUGATE OF COMPLEX NUMBER

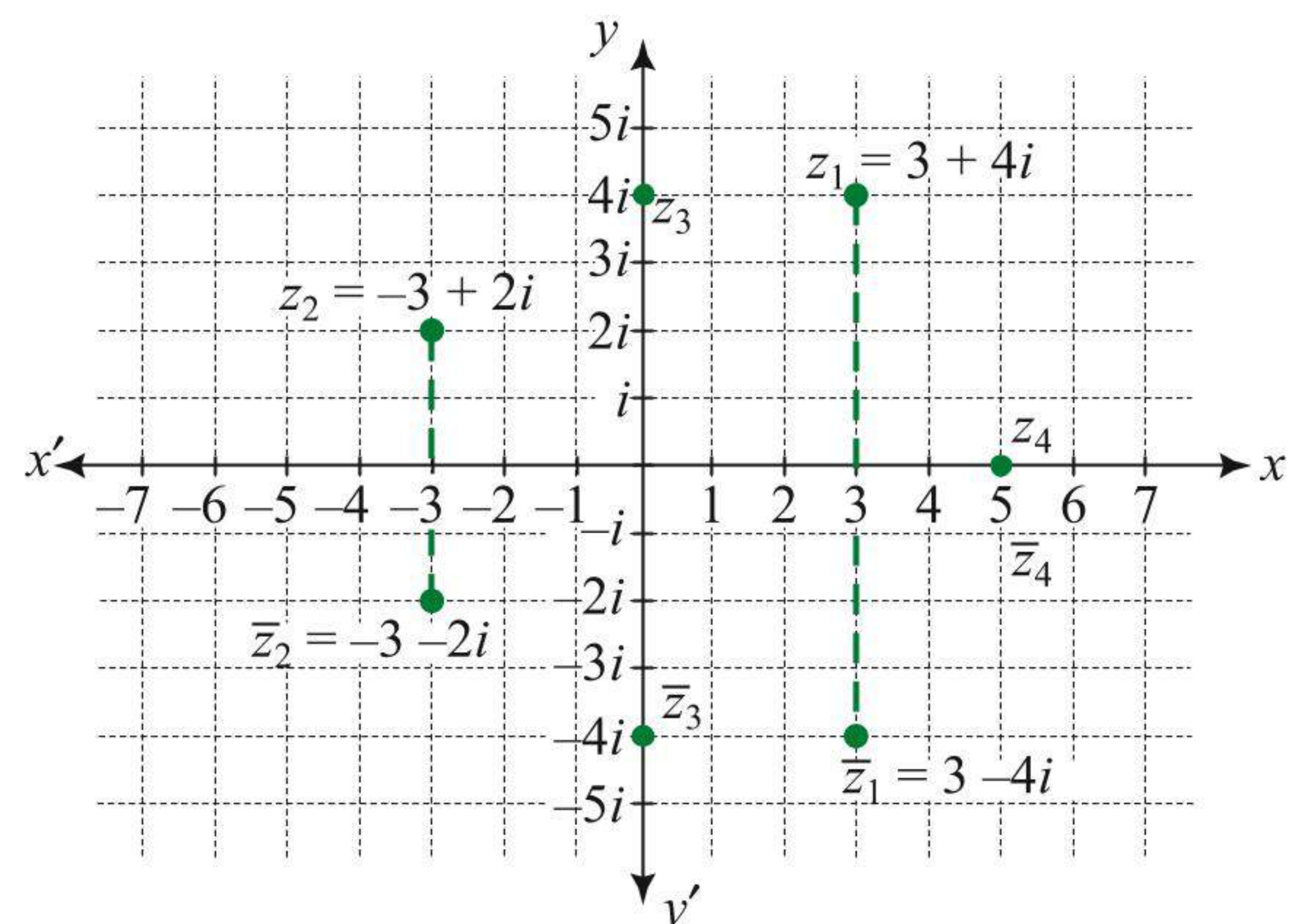
Conjugate of complex number $z = a + ib$ is defined as $\bar{z} = a - ib$.

e.g., for $z = 3 + 5i$, $\bar{z} = 3 - 5i$

for $z = 2 - 3i$, $\bar{z} = 2 + 3i$

for $z = 4i$, $\bar{z} = -4i$.

Thus, on complex plane for ordered pair (a, b) its conjugate is $(a, -b)$. Clearly, $(a, -b)$ is the reflection of (a, b) in real axis.

**Properties of Conjugate**

- (i) $\overline{(\bar{z})} = z$
- (ii) $z + \bar{z} = 2\text{Re}(z)$
- (iii) $z - \bar{z} = 2i \text{Im}(z)$
- (iv) $z = \bar{z} \Leftrightarrow z$ is purely real
- (v) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary
- (vi) $z\bar{z} = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2$
- (vii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- (viii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

$$(ix) \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

From this we can say that $\overline{z^n} = \bar{z}^n$, when $n \in \mathbb{N}$

$$(x) \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$$

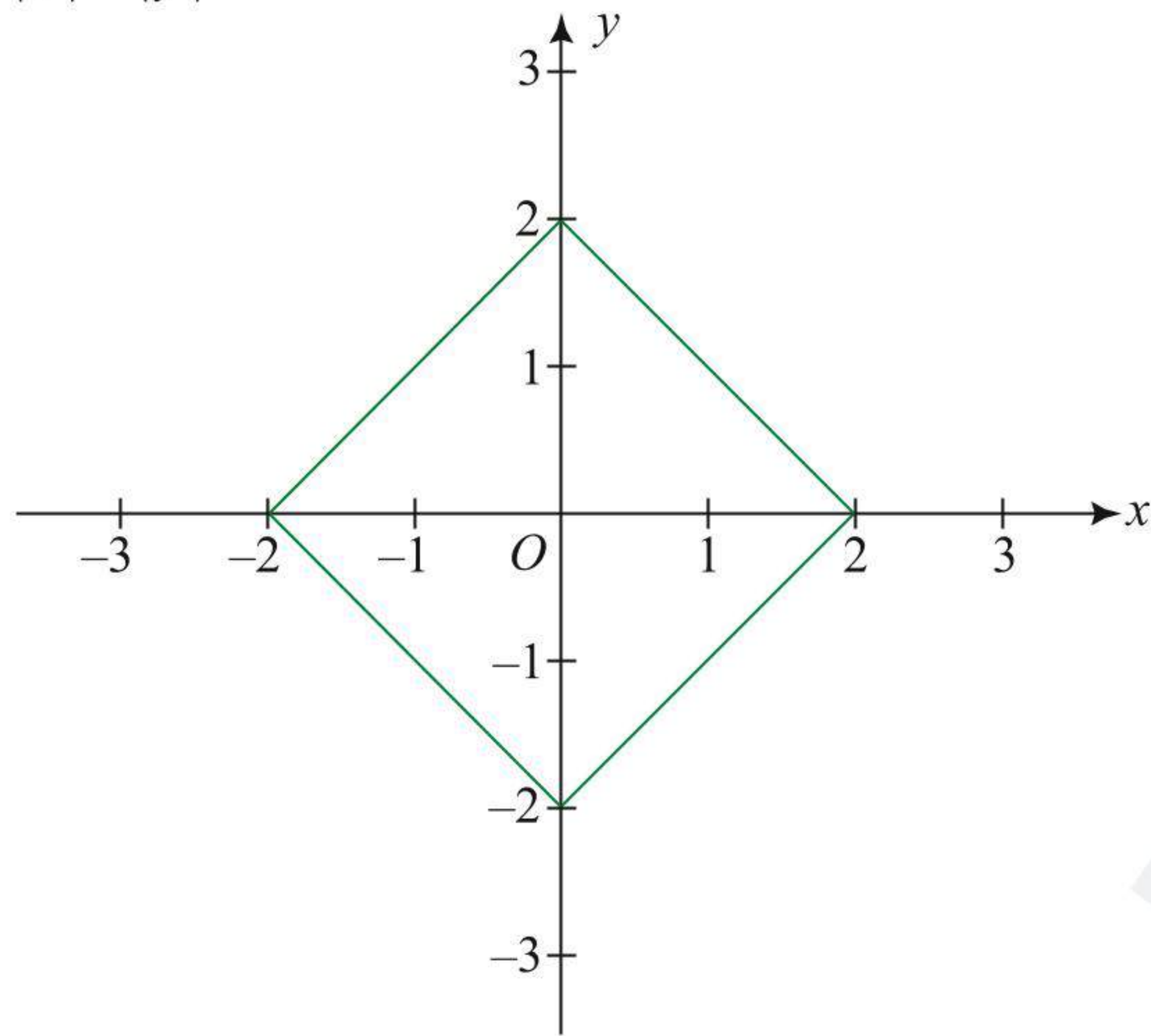
ILLUSTRATION 3.14

If z is a complex number such that $|z - \bar{z}| + |z + \bar{z}| = 4$, then find the area bounded by the locus of z .

Sol. We have $|z - \bar{z}| + |z + \bar{z}| = 4$

$$\therefore |2iy| + |2x| = 4$$

$$\Rightarrow |x| + |y| = 2$$



This equation forms square with vertices $(\pm 2, 0)$ and $(0, \pm 2)$.

So, area of square is 8 sq. units.

ILLUSTRATION 3.15

If $(x + iy)^5 = p + iq$, then prove that $(y + ix)^5 = q + ip$.

Sol. $(x + iy)^5 = p + iq$

$$\text{or } \overline{(x + iy)^5} = \overline{p + iq}$$

$$\text{or } \overline{(x + iy)^5} = p - iq$$

$$\text{or } (x - iy)^5 = p - iq$$

$$\text{or } i^5(x - iy)^5 = pi^5 - i^6 q$$

$$\text{or } (xi - i^2 y)^5 = pi + q$$

$$\text{or } (y + ix)^5 = pi + q$$

ILLUSTRATION 3.16

If $z = x + iy$ lies in the third quadrant, then prove that $\frac{\bar{z}}{z}$ also lies in the third quadrant when $y < x < 0$.

Sol. Given that, $z = x + iy$ lies in third quadrant.

$$\Rightarrow x < 0 \text{ and } y < 0$$

$$\text{Now, } \frac{\bar{z}}{z} = \frac{x - iy}{x + iy} = \frac{(x - iy)(x - iy)}{(x + iy)(x - iy)}$$

$$= \frac{x^2 - y^2 - 2ixy}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2} - \frac{2ixy}{x^2 + y^2}$$

For $\frac{\bar{z}}{z}$ to lie in the third quadrant, we must have

$$\frac{x^2 - y^2}{x^2 + y^2} < 0 \text{ and } \frac{-2xy}{x^2 + y^2} < 0$$

$$\Rightarrow x^2 - y^2 < 0 \text{ and } -2xy < 0$$

$$\Rightarrow x^2 < y^2 \text{ and } xy > 0$$

$$\text{But } x, y < 0$$

$$\therefore y < x < 0$$

ILLUSTRATION 3.17

Prove that, $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$ is purely real.

Sol.

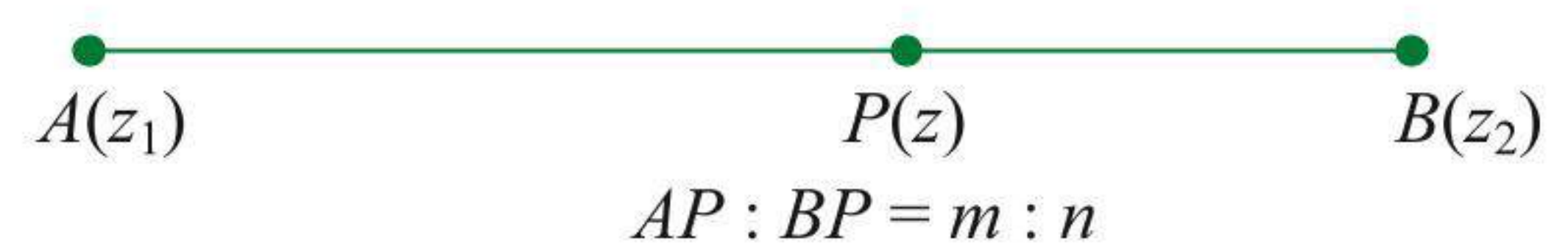
$$\begin{aligned} \text{We have, } & \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5 \\ &= \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \overline{\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5} \\ &= \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \overline{\left(\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5\right)} \quad \left(\because (\bar{z})^n = \overline{z^n}\right) \\ &= z + \bar{z}, \text{ where } z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 \\ &= 2\text{Re}(z) \end{aligned}$$

Hence, given complex number is purely real.

SECTION FORMULA IN COMPLEX NUMBERS

Consider two points A and B in the argand plane having complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

$$\therefore A \equiv (x_1, y_1) \text{ and } B \equiv (x_2, y_2).$$



Coordinates of point P dividing AB internally in the ratio $m : n$ are

$$\text{given by } \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right).$$

$\therefore$ Complex number of point P is,

$$\begin{aligned} z &= \frac{mx_2 + nx_1}{m+n} + i \left(\frac{my_2 + ny_1}{m+n}\right) \\ &= \frac{m(x_2 + iy_2) + n(x_1 + iy_1)}{m+n} = \frac{mz_2 + nz_1}{m+n} \end{aligned}$$

If point $P(z)$ divides AB externally in the ratio $m : n$ then complex number of P is given by $\frac{mz_2 - nz_1}{m-n}$.



- Complex number of mid-point of line segment joining z_1 and z_2 is $\frac{z_1 + z_2}{2}$.
- Centroid of the triangle formed by points $A(z_1)$, $B(z_2)$ and $C(z_3)$ is given by complex number $\frac{z_1 + z_2 + z_3}{3}$.
- Incentre of the triangle formed by points $A(z_1)$, $B(z_2)$ and $C(z_3)$ is given by complex number $\frac{(BC)z_1 + (AC)z_2 + (AB)z_3}{AB + BC + AC}$.
- In acute triangle orthocentre (H), centroid (G) and circumcentre (O) are collinear and $HG : GO \equiv 2 : 1$.
So, if circumcentre of the triangle formed by points $A(z_1)$, $B(z_2)$ and $C(z_3)$ is origin then its orthocentre is $(z_1 + z_2 + z_3)$.

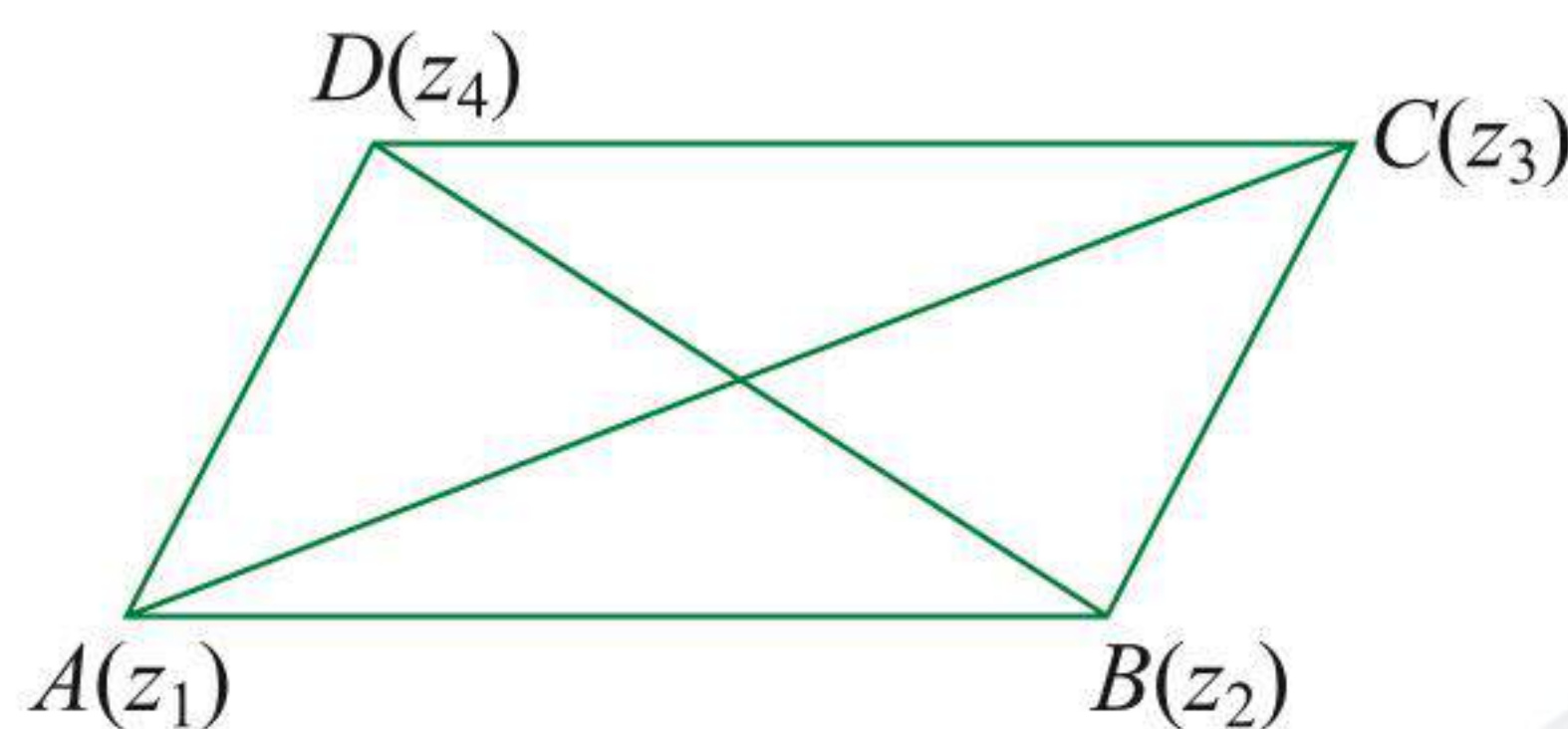
ILLUSTRATION 3.18

Find the relation if z_1, z_2, z_3 and z_4 are the affixes of the vertices of a parallelogram taken in order.

Sol. As the diagonals of a parallelogram bisect each other, therefore, complex number of the mid-point of AC is same as the complex number of the mid-point of BD .

$$\therefore \frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$$

$$\text{or } z_1 + z_3 = z_2 + z_4$$

**ILLUSTRATION 3.19**

Let z_1, z_2, z_3 be three complex numbers and a, b, c be real numbers not all zero, such that $a + b + c = 0$ and $az_1 + bz_2 + cz_3 = 0$. Show that z_1, z_2, z_3 are collinear.

Sol. Given,

$$a + b + c = 0 \quad (1)$$

$$\text{and } az_1 + bz_2 + cz_3 = 0 \quad (2)$$

Since a, b, c are not all zero, from (2), we have

$$az_1 + bz_2 - (a + b)z_3 = 0 \quad [\text{From (1), } c = -(a + b)]$$

$$\text{or } az_1 + bz_2 = (a + b)z_3$$

$$\text{or } z_3 = \frac{az_1 + bz_2}{a + b} \quad (3)$$

From (3), it follows that z_3 divides the line segment joining z_1 and z_2 internally in the ratio $b:a$.

Hence, z_1, z_2 and z_3 are collinear.

If a and b are of the same sign, then division is in fact internal, and if a and b are of opposite sign, then division is external in the ratio $|b|:|a|$.

CONCEPT APPLICATION EXERCISE 3.2

- Find the multiplicative inverse of the complex number $4 - 3i$.
- Express the following complex numbers in $a + ib$ form:
 - $\frac{(3 - 2i)(2 + 3i)}{(1 + 2i)(2 - i)}$
 - $\frac{2 - \sqrt{-25}}{1 - \sqrt{-16}}$

- Find the least positive integer n such that $\left(\frac{2i}{1+i}\right)^n$ is a positive integer.
- If one root of the equation $z^2 - az + a - 1 = 0$ is $(1 + i)$, where a is a complex number, then find the other root.
- Prove that quadrilateral formed by the complex numbers which are roots of the equation $z^4 - z^3 + 2z^2 - z + 1 = 0$ is an equilateral trapezium.
- If Z is a non-real complex number, then find the minimum value of $\frac{\text{Im } Z^5}{(\text{Im } Z)^5}$.
- Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.
- If z_1, z_2, z_3 are three nonzero complex numbers such that $z_3 = (1 - \lambda)z_1 + \lambda z_2$ where $\lambda \in \mathbb{R} - \{0\}$, then prove that points corresponding to z_1, z_2 , and z_3 are collinear.
- Prove that for positive integers n_1 and n_2 , the value of expression $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$, where $i = \sqrt{-1}$, is a real number.

ANSWERS

- $\frac{4}{25} + \frac{3}{25}i$
- (a) $\frac{63}{25} - \frac{16}{25}i$ (b) $\frac{12}{17} + \frac{3}{17}i$
- $n = 8$
- $z = 1$
- -4
- $x = 3, y = -3$

EQUALITY OF COMPLEX NUMBERS

We know that two ordered pairs (a, b) and (c, d) are same if their corresponding first elements are same and corresponding second elements are also same, i.e., $a = c$ and $b = d$.

Complex numbers $z_1 = a + ib$ and $z_2 = c + id$, are also expressed as ordered pairs (a, b) and (c, d) , respectively. So, complex numbers z_1 and z_2 are same if $a = c$ and $b = d$, i.e., their real parts are same and their imaginary parts are same.

Thus, $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

We know that $(a, b) > (c, d)$ is absurd, as there is no mathematical or geometrical interpretation of it.

Similarly, in complex numbers inequality has no meaning, i.e., $a + ib > c + id$ is meaningless.

$$a + ib = 0 \Leftrightarrow a = 0 \text{ and } b = 0.$$

ILLUSTRATION 3.20

Find real values of x and y for which the complex numbers $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugate of each other.

Sol. Given, $-3 + ix^2y = \overline{x^2 + y + 4i}$

$$\text{or } -3 + ix^2y = x^2 + y - 4i$$

$$\text{or } -3 = x^2 + y \quad (1)$$

$$\text{and } x^2y = -4 \quad (2)$$

$$\therefore -3 = x^2 - \frac{4}{x^2} \quad [\text{Putting } y = -4/x^2 \text{ from (2) in (1)}]$$

$$\text{or } x^4 + 3x^2 - 4 = 0$$

$$\begin{aligned}\text{or } (x^2 + 4)(x^2 - 1) &= 0 \\ \text{or } x^2 - 1 &= 0 \\ \text{or } x &= \pm 1\end{aligned}$$

From (2), $y = -4$, when $x = \pm 1$. Hence, $x = 1, y = -4$ or $x = -1, y = -4$.

ILLUSTRATION 3.21

Given that $x, y \in R$. Solve: $\frac{x}{1+2i} + \frac{y}{3+2i} = \frac{5+6i}{8i-1}$

$$\begin{aligned}\text{Sol. } \frac{x}{1+2i} + \frac{y}{3+2i} &= \frac{5+6i}{8i-1} \\ \text{or } \frac{x(1-2i)}{1-4i^2} + \frac{y(3-2i)}{9-4i^2} &= \frac{(5+6i)(8i+1)}{(8i)^2 - 1^2} \\ \text{or } \frac{x-2xi}{5} + \frac{3y-2yi}{13} &= \frac{40i+5-48+6i}{-64-1} \\ \text{or } \frac{13x-26xi+15y-10yi}{65} &= \frac{-43+46i}{-65} \\ \text{or } (13x+15y) - i(26x+10y) &= 43-46i \\ \text{Equating real and imaginary parts, we get} \\ 13x+15y &= 43 \quad (1) \\ 13x+5y &= 23 \quad (2)\end{aligned}$$

Solving for x and y , we get $x = 1$ and $y = 2$

ILLUSTRATION 3.22

If $(x+iy)^3 = u+iv$, then show that $\frac{u}{x} + \frac{v}{y} = (x^2 - y^2)$.

$$\begin{aligned}\text{Sol. } (x+iy)^3 &= u+iv \\ \text{or } x^3 + (iy)^3 + 3 \cdot x \cdot iy(x+iy) &= u+iv \\ \text{or } x^3 + i^3y^3 + 3x^2yi + 3xy^2i^2 &= u+iv \\ \text{or } x^3 - iy^3 + 3x^2yi - 3xy^2 &= u+iv \\ \text{or } (x^3 - 3xy^2) + i(3x^2y - y^3) &= u+iv \\ \text{On equating real and imaginary parts, we get} \\ u = x^3 - 3xy^2, v = 3x^2y - y^3 \\ \therefore \frac{u}{x} + \frac{v}{y} &= \frac{x^3 - 3xy^2}{x} + \frac{3x^2y - y^3}{y} \\ &= \frac{x(x^2 - 3y^2)}{x} + \frac{y(3x^2 - y^2)}{y} \\ &= x^2 - 3y^2 + 3x^2 - y^2 \\ &= 4x^2 - 4y^2 = 4(x^2 - y^2)\end{aligned}$$

ILLUSTRATION 3.23

Let z be a complex number satisfying the equation $z^2 - (3+i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root. Then find the nonreal root.

$$\begin{aligned}\text{Sol. } \text{Let } \alpha &\text{ be the real root. Then,} \\ \alpha^2 - (3+i)\alpha + m + 2i &= 0 \\ \Rightarrow (\alpha^2 - 3\alpha + m) + i(2 - \alpha) &= 0 \\ \Rightarrow \alpha^2 - 3\alpha + m = 0 \text{ and } 2 - \alpha &= 0 \\ \Rightarrow \alpha = 2 \text{ and } m = 2\end{aligned}$$

Product of the roots is $2(1+i)$ with one root as 2. Hence, the nonreal root is $1+i$.

ILLUSTRATION 3.24

Show that the equation $Z^4 + 2Z^3 + 3Z^2 + 4Z + 5 = 0$ has no root which is either purely real or purely imaginary.

$$\text{Sol. } Z^4 + 2Z^3 + 3Z^2 + 4Z + 5 = 0$$

If α is a purely real root of the above equation, then

$$\begin{aligned}\alpha^4 + 2\alpha^3 + 3\alpha^2 + 4\alpha + 5 &= 0 \\ \therefore (\alpha^2 + \alpha)^2 + 2(\alpha + 1)^2 + 3 &= 0 \text{ which is not possible as all} \\ \text{the terms of LHS are positive.}\end{aligned}$$

If $i\beta$ is a purely imaginary root of the given equation, then

$$\begin{aligned}\beta^4 - 2i\beta^3 - 3\beta^2 + 4i\beta + 5 &= 0 \\ \therefore \beta^4 - 3\beta^2 + 5 = 0 \text{ and } 2\beta^3 - 4\beta &= 0 \\ \text{i.e., } \left(\beta^2 - \frac{3}{2}\right)^2 + \frac{11}{4} &= 0 \text{ and } 2\beta(\beta^2 - 2) = 0\end{aligned}$$

So, no value of β exists.

Hence, equation has neither purely real nor purely imaginary roots.

SQUARE ROOT OF COMPLEX NUMBER

Let $a+ib$ be a complex number such that $\sqrt{a+ib} = x+iy$, where x and y are real numbers. Then,

$$\begin{aligned}\sqrt{a+ib} &= x+iy \\ \text{or } (a+ib) &= (x+iy)^2 \\ \text{or } a+ib &= (x^2 - y^2) + 2ixy \\ \text{On equating real and imaginary parts, we get} \\ x^2 - y^2 &= a \quad (1)\end{aligned}$$

$$\begin{aligned}\text{and } 2xy &= b \\ \text{Now, } (x^2 + y^2)^2 &= (x^2 - y^2)^2 + 4x^2y^2 \\ \text{or } (x^2 + y^2)^2 &= a^2 + b^2 \\ \text{or } (x^2 + y^2) &= \sqrt{a^2 + b^2} \quad [\because x^2 + y^2 \geq 0] \quad (2)\end{aligned}$$

Solving (1) and (2), we get

$$\begin{aligned}x^2 &= \left(\frac{1}{2}\right) \left[\sqrt{a^2 + b^2} + a \right] \\ \text{and } y^2 &= \left(\frac{1}{2}\right) \left[\sqrt{a^2 + b^2} - a \right] \\ \Rightarrow x &= \pm \sqrt{\left(\frac{1}{2}\right) \left[\sqrt{a^2 + b^2} + a \right]} \\ \text{and } y &= \pm \sqrt{\left(\frac{1}{2}\right) \left[\sqrt{a^2 + b^2} - a \right]}\end{aligned}$$

If b is positive, then by the relation $2xy = b$, x and y are of the same sign. Hence,

$$\sqrt{a+ib} = \pm \left\{ \sqrt{\frac{1}{2} \left[\sqrt{a^2 + b^2} + a \right]} + i \sqrt{\frac{1}{2} \left[\sqrt{a^2 + b^2} - a \right]} \right\}$$

If b is negative, then by the relation $2xy = b$, x and y are of different signs. Hence,

$$\sqrt{a+ib} = \pm \left\{ \sqrt{\frac{1}{2} \left[\sqrt{a^2 + b^2} + a \right]} - i \sqrt{\frac{1}{2} \left[\sqrt{a^2 + b^2} - a \right]} \right\}$$

ILLUSTRATION 3.25

Find the square roots of the following:

- (i)
- $7 - 24i$
- (ii)
- $5 + 12i$
- (iii)
- $-15 - 8i$

Sol.

- (i) Let
- $\sqrt{7 - 24i} = x + iy$
- . Then,

$$\sqrt{7 - 24i} = x + iy$$

$$\text{or } 7 - 24i = (x + iy)^2$$

$$\text{or } 7 - 24i = (x^2 - y^2) + 2ixy$$

$$\text{or } x^2 - y^2 = 7 \quad (1)$$

$$\text{and } 2xy = -24 \quad (2)$$

$$\text{Now, } (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$$

$$\text{or } (x^2 + y^2)^2 = 49 + 576 = 625$$

$$\text{or } x^2 + y^2 = 25 \quad [\because x^2 + y^2 > 0] \quad (3)$$

On solving (1) and (3), we get

$$x^2 = 16 \text{ and } y^2 = 9 \Rightarrow x = \pm 4 \text{ and } y = \pm 3$$

From (2), $2xy$ is negative. So, x and y are of opposite signs.Hence, $x = 4$ and $y = -3$ or $x = -4$ and $y = 3$.

$$\text{Hence, } \sqrt{7 - 24i} = \pm(4 - 3i).$$

- (ii) Let
- $\sqrt{5 + 12i} = x + iy$
- . Then,

$$\sqrt{5 + 12i} = x + iy$$

$$\text{or } 5 + 12i = (x + iy)^2$$

$$\text{or } 5 + 12i = (x^2 - y^2) + 2ixy$$

$$\text{or } x^2 - y^2 = 5 \quad (1)$$

$$\text{and } 2xy = 12 \quad (2)$$

$$\text{Now, } (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$$

$$\text{or } (x^2 + y^2)^2 = 5^2 + 12^2 = 169$$

$$\text{or } x^2 + y^2 = 13 \quad (\because x^2 + y^2 > 0) \quad (3)$$

On solving (1) and (3), we get

$$x^2 = 9 \text{ and } y^2 = 4 \Rightarrow x = \pm 3 \text{ and } y = \pm 2$$

From (2), $2xy$ is positive. So, x and y are of the same sign.

Hence,

$$x = 3 \text{ and } y = 2 \text{ or } x = -3 \text{ and } y = -2$$

$$\text{Hence, } \sqrt{5 + 12i} = \pm(3 + 2i).$$

- (iii) Let
- $\sqrt{-15 - 8i} = x + iy$
- . Then,

$$\sqrt{-15 - 8i} = x + iy$$

$$\text{or } -15 - 8i = (x + iy)^2$$

$$\text{or } -15 - 8i = (x^2 - y^2) + 2ixy$$

$$\text{or } -15 = x^2 - y^2 \quad (1)$$

$$\text{and } 2xy = -8 \quad (2)$$

$$\text{Now, } (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$$

$$\text{or } (x^2 + y^2)^2 = (-15)^2 + 64 = 289$$

$$\text{or } x^2 + y^2 = 17$$

On solving (1) and (3), we get

$$x^2 = 1 \text{ and } y^2 = 16 \Rightarrow x = \pm 1 \text{ and } y = \pm 4$$

From (2), $2xy$ is negative. So, x and y are of opposite signs.Hence, $x = 1$ and $y = -4$ or $x = -1$ and $y = 4$

$$\text{Hence, } \sqrt{-15 - 8i} = \pm(1 - 4i).$$

ILLUSTRATION 3.26Find all possible values of $\sqrt{i} + \sqrt{-i}$.

$$\text{Sol. } \sqrt{i} + \sqrt{-i} = \sqrt{0 + 1 \cdot i} + \sqrt{0 - 1 \cdot i}$$

$$\text{Now } \sqrt{a + i \cdot b} = \pm \left\{ \sqrt{\frac{1}{2} \{ \sqrt{a^2 + b^2} + a \}} + i \sqrt{\frac{1}{2} \{ \sqrt{a^2 + b^2} - a \}} \right\}$$

$$\text{and } \sqrt{a - i \cdot b} = \pm \left\{ \sqrt{\frac{1}{2} \{ \sqrt{a^2 + b^2} + a \}} - i \sqrt{\frac{1}{2} \{ \sqrt{a^2 + b^2} - a \}} \right\}$$

$$\Rightarrow \sqrt{0 + 1 \cdot i} = \pm \left\{ \sqrt{\frac{1}{2} \{ \sqrt{0 + 1^2} + 0 \}} + i \sqrt{\frac{1}{2} \{ \sqrt{0 + 1^2} - 0 \}} \right\}$$

$$= \pm \frac{1}{\sqrt{2}}(1 + i)$$

$$\text{and } \sqrt{0 - 1 \cdot i} = \pm \left\{ \sqrt{\frac{1}{2} \{ \sqrt{0 + 1^2} + 0 \}} - i \sqrt{\frac{1}{2} \{ \sqrt{0 + 1^2} - 0 \}} \right\}$$

$$= \pm \frac{1}{\sqrt{2}}(1 - i)$$

$$\text{Now, } \sqrt{i} + \sqrt{-i} = \pm \frac{1}{\sqrt{2}}(1 + i) \pm \frac{1}{\sqrt{2}}(1 - i)$$

$$\text{or } \sqrt{i} + \sqrt{-i} = \pm \sqrt{2} + 0 \cdot i \text{ or } 0 \pm \sqrt{2}i$$

ILLUSTRATION 3.27Solve for z : $z^2 - (3 - 2i)z = (5i - 5)$.**Sol.**

$$z^2 - (3 - 2i)z = (5i - 5)$$

$$\text{or } z^2 - (3 - 2i)z - (5i - 5) = 0$$

$$\text{or } z = \frac{(3 - 2i) \pm \sqrt{(3 - 2i)^2 + 4(5i - 5)}}{2}$$

$$= \frac{(3 - 2i) \pm \sqrt{9 - 4 - 12i + 20i - 20}}{2}$$

$$= \frac{(3 - 2i) \pm \sqrt{8i - 15}}{2}$$

$$\text{Now } \sqrt{-15 + 8i}$$

$$= \pm \left\{ \sqrt{\frac{1}{2}(\sqrt{289} - 15)} + \sqrt{\frac{1}{2}(\sqrt{289} + 15)} \right\}$$

$$= \pm(1 + 4i)$$

$$\Rightarrow z = \frac{3 - 2i \pm (1 + 4i)}{2}$$

$$\Rightarrow z = (2 + i) \text{ and } (1 - 3i)$$

CONCEPT APPLICATION EXERCISE 3.3

1. If $(a + b) - i(3a + 2b) = 5 + 2i$, then find a and b .
2. Find all nonzero complex numbers z satisfying $\bar{z} = iz^2$.
3. If a, b, c are nonzero real numbers and $az^2 + bz + c + i = 0$ has purely imaginary roots, then prove that $a = b^2c$.
4. If the sum of square of roots of the equation $x^2 + (p + iq)x + 3i = 0$ is 8, then find the values of p and q , where p and q are real.
5. Find the square root of $9 + 40i$.
6. Simplify $\frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} - \sqrt{5-12i}}$.
7. If $\sqrt{x + iy} = \pm(a + ib)$, then find $\sqrt{-x - iy}$.

ANSWERS

1. $a = -12, b = 17$
2. $z = i, \pm \frac{\sqrt{3}}{2} - \frac{i}{2}$
4. $p = 3, q = 1$ or $p = -3, q = -1$
5. $(5 + 4i)$ or $-(5 + 4i)$
6. $-\frac{3}{2}i$
7. $\pm(b - ia)$

CUBE ROOTS OF UNITY

Let $z = 1^{1/3}$. Then,
 $z^3 = 1$

or $z^3 - 1 = 0$

or $(z - 1)(z^2 + z + 1) = 0$

$\Rightarrow z - 1 = 0$ or $z^2 + z + 1 = 0$

$\Rightarrow z = 1$ or $z = \frac{-1 \pm \sqrt{1-4}}{2}$

$\Rightarrow z = 1$ or $z = \frac{-1 \pm i\sqrt{3}}{2}$

So, the cube roots of unity are $1, (-1 + i\sqrt{3})/2$ and $(-1 - i\sqrt{3})/2$. Clearly, one of the roots of unity is real and the other two are complex.

PROPERTIES OF CUBE ROOTS OF UNITY

1. Each complex cube root of unity is the square of the other.

Proof:

Complex cube roots of unity are $(-1 + i\sqrt{3})/2$ and $(-1 - i\sqrt{3})/2$.

Now,

$$\left(\frac{-1 + i\sqrt{3}}{2}\right)^2 = \frac{1 - 2i\sqrt{3} + 3i^2}{4} = \frac{1 - 2i\sqrt{3} - 3}{4} = \left(\frac{-1 - i\sqrt{3}}{2}\right)$$

$$\left(\frac{-1 - i\sqrt{3}}{2}\right)^2 = \frac{1 + 2i\sqrt{3} + 3i^2}{4} = \frac{1 + 2i\sqrt{3} - 3}{4} = \left(\frac{-1 + i\sqrt{3}}{2}\right)$$

Hence, each complex cube root of unity is the square of the other.

It follows from the above discussion that if we denote one of the complex cube roots of unity by ω (omega), then the other complex cube root of unity is ω^2 .

Let $\omega = (-1 + i\sqrt{3})/2$

Then $\omega^2 = (-1 - i\sqrt{3})/2$

Clearly, $\bar{\omega} = \omega^2$ and $\overline{\omega^2} = \omega$.

2. Integral powers of ω :

Since ω is a root of the equation $z^3 - 1 = 0$, it satisfies the equation $z^3 - 1 = 0$. Therefore,

$$\omega^3 - 1 = 0 \text{ or } \omega^3 = 1$$

Since $\omega^3 = 1$, we have $\omega^n = \omega^r$, where r is the least non-negative remainder obtained by dividing n by 3. For example, $\omega^{18} = (\omega^3)^6 = 1^6 = 1$, $\omega^{20} = (\omega^3)^6 \omega^2 = 1^6 \omega^2 = \omega^2$, $\omega^{-30} = (\omega^3)^{-10} = 1^{-10} = 1$, $\omega^{28} = (\omega^{27}) \omega = \omega$.

3. The sum of three cube roots of unity is zero, i.e.,

$$1 + \omega + \omega^2 = 0.$$

Proof:

$$\begin{aligned} \text{We have, } 1 + \omega + \omega^2 &= 1 + \left(\frac{-1 + i\sqrt{3}}{2}\right) + \left(\frac{-1 - i\sqrt{3}}{2}\right) \\ &= \frac{2 - 1 + i\sqrt{3} - 1 - i\sqrt{3}}{2} = 0 \end{aligned}$$

4. The product of three cube roots of unity is 1.

Proof:

Three cube roots of unity are $1, \omega$, and ω^2 . So, the product of cube roots of unity is $1 \times \omega \times \omega^2 = \omega^3 = 1$.

5. Each complex cube root of unity is the reciprocal of the other.

Proof:

We have,

$$\omega \times \omega^2 = \omega^3 = 1 \Rightarrow \omega = \frac{1}{\omega^2} \text{ and } \omega^2 = \frac{1}{\omega}$$

6. Cube roots of -1 are $-1, -\omega$, and $-\omega^2$, where ω is complex cube root of unity.

Proof:

$$z = (-1)^{1/3}$$

$$\text{or } z^3 = -1$$

$$\text{or } (-z)^3 = 1$$

$$\text{or } -z = 1, \omega, \omega^2$$

$$\text{or } z = -1, -\omega, -\omega^2$$

The idea of finding cube roots of 1 and -1 can be extended to find cube roots of any real number. If a is any positive real number, then $a^{1/3}$ has values $a^{1/3}, a^{1/3}\omega$, and $a^{1/3}\omega^2$. If a is a negative real number, then $a^{1/3}$ has values $-|a|^{1/3}, -|a|^{1/3}\omega$, and $-|a|^{1/3}\omega^2$. For example, $8^{1/3}$ has values $2, 2\omega$, and $2\omega^2$ whereas $(-8)^{1/3}$ attains values $-2, -2\omega$, and $-2\omega^2$.

7. If $1, \omega, \omega^2$ are cube roots of unity and n is a positive integer,

$$\text{then } 1 + \omega^n + \omega^{2n} = \begin{cases} 3, & \text{when } n \text{ is a multiple of 3} \\ 0, & \text{when } n \text{ is not a multiple of 3} \end{cases}$$

Proof:

Case I: n is a multiple of 3.

In this case, $n = 3m$ for some positive integer m . Thus,

$$\begin{aligned} 1 + \omega^n + \omega^{2n} &= 1 + \omega^{3m} + \omega^{6m} \\ &= 1 + (\omega^3)^m + (\omega^3)^{2m} = 1 + 1 + 1 = 3 \\ [\because (\omega^3)^m &= 1^m = 1 \text{ and } (\omega^3)^{2m} = 1^{2m} = 1] \end{aligned}$$

Case II: n is not a multiple of 3.

In this case, $n = 3m + 1$ or $n = 3m + 2$ for some positive integer m . When $n = 3m + 1$,

$$\begin{aligned} 1 + \omega^n + \omega^{2n} &= 1 + \omega^{3m+1} + \omega^{6m+2} \\ &= 1 + \omega^{3m}\omega + \omega^{6m}\omega^2 \\ &= 1 + (\omega^3)^m\omega + (\omega^3)^{2m}\omega^2 \\ &= 1 + \omega + \omega^2 = 0 \end{aligned}$$

When $n = 3m + 2$,

$$\begin{aligned} 1 + \omega^n + \omega^{2n} &= 1 + \omega^{3m+2} + \omega^{6m+4} \\ &= 1 + \omega^{3m}\omega^2 + \omega^{6m}\omega^4 \\ &= 1 + (\omega^3)^m\omega^2 + (\omega^3)^{2m}\omega^3\omega \\ &= 1 + \omega^2 + \omega = 0 \end{aligned}$$

8. Factorization of $a^3 + b^3$ and $a^3 - b^3$:

$$\begin{aligned} a^3 + b^3 &= (a + b)(a^2 - ab + b^2) \\ &= (a + b)(a + b\omega)(a + b\omega^2) \end{aligned}$$

$$\begin{aligned} \text{and } a^3 - b^3 &= (a - b)(a^2 + ab + b^2) \\ &= (a - b)(a - b\omega)(a - b\omega^2) \end{aligned}$$

9. Factorization of $a^3 + b^3 + c^3 - 3abc$

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) \end{aligned}$$

10. Cube roots of unity represent vertices of equilateral triangle on the Argand plane.

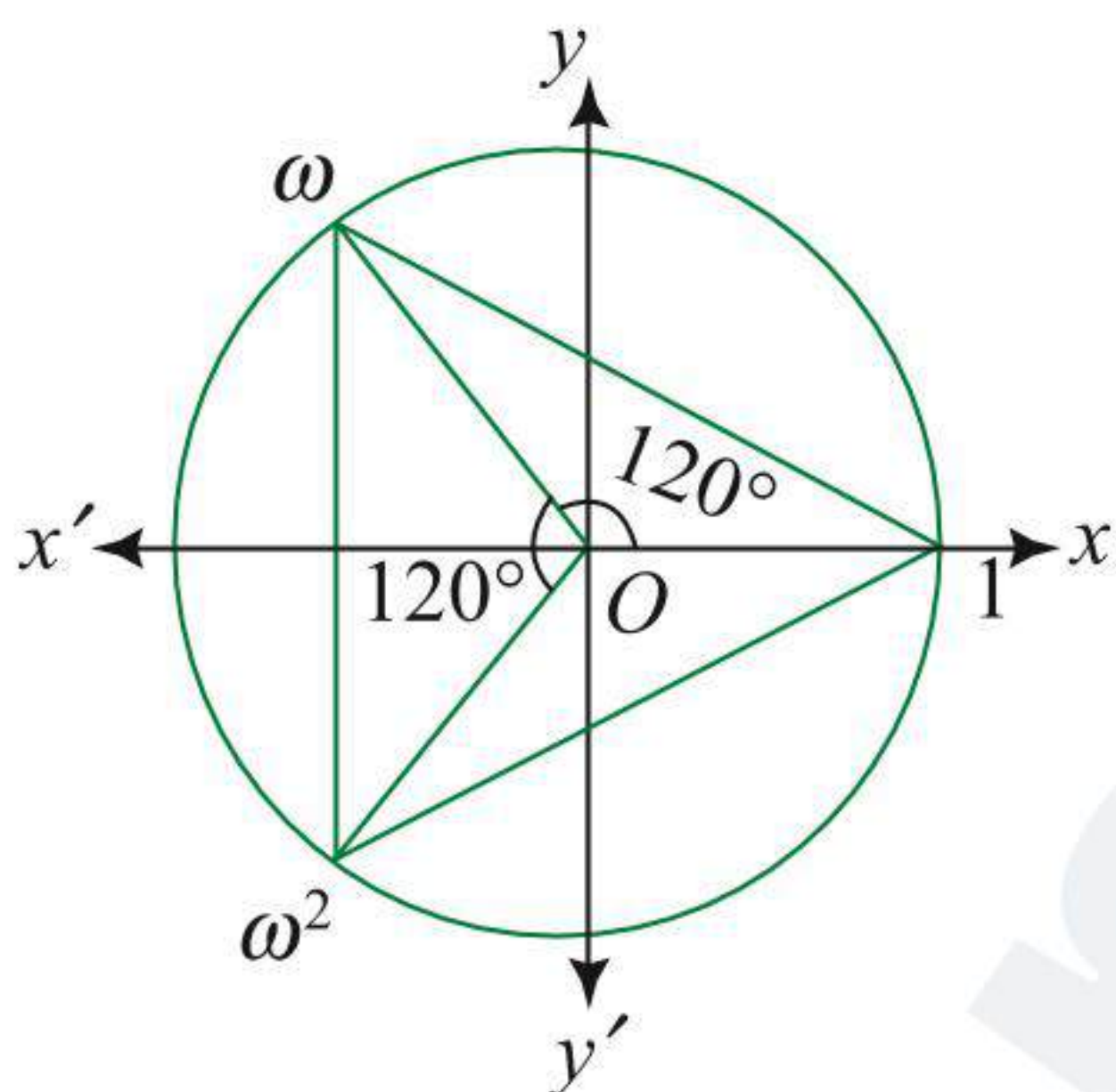


ILLUSTRATION 3.28

Solve the equation $(x - 1)^3 + 8 = 0$ in the set C of all complex numbers.

Sol. We have,

$$(x - 1)^3 + 8 = 0 \quad \text{or } (x - 1)^3 = -8 \quad \text{or } x - 1 = (-8)^{1/3}$$

$$\text{or } x - 1 = 2(-1)^{1/3}$$

$$\begin{aligned} \text{or } x - 1 &= 2(-1) \quad \text{or } x - 1 = 2(-\omega) \quad \text{or } x - 1 = 2(-\omega^2) \\ &[\because (-1)^{1/3} = -1 \text{ or } -\omega \text{ or } -\omega^2] \end{aligned}$$

$$x - 1 = -2 \quad \text{or } x - 1 = -2\omega \quad \text{or } x - 1 = -2\omega^2$$

$$\text{or } x = -1 \quad \text{or } x = 1 - 2\omega \quad \text{or } x = 1 - 2\omega^2$$

Hence, the solutions of the equation $(x - 1)^3 + 8 = 0$ are -1 , $1 - 2\omega$, and $1 - 2\omega^2$.

ILLUSTRATION 3.29

If n is an odd integer that is greater than or equal to 3 but not a multiple of 3, then prove that $(x + 1)^n - x^n - 1$ is divisible by $x^3 + x^2 + x$.

Sol. Let $f(x) = (x + 1)^n - x^n - 1$

$$x^3 + x^2 + x = x(x^2 + x + 1) = x(x - \omega)(x - \omega^2)$$

$$f(0) = (0 + 1)^n - 0^n - 1 = 0$$

$$f(\omega) = (\omega + 1)^n - \omega^n - 1 = (-\omega^2)^n - \omega^n - 1 = -(\omega^{2n} + \omega^n + 1) = 0$$

when n is not a multiple of 3.

$$f(\omega^2) = (\omega^2 + 1)^n - \omega^{2n} - 1$$

$$= (-\omega)^n - \omega^{2n} - 1 = -(\omega^n + \omega^{2n} + 1) = 0$$

when n is not a multiple of 3.

ILLUSTRATION 3.30

ω is an imaginary root of unity.

Prove that

$$\begin{aligned} \text{(i)} \quad (a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 &= (2a - b - c)(2b - a - c)(2c - a - b). \end{aligned}$$

(ii) If $a + b + c = 0$, then prove that

$$(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 = 27abc.$$

Sol.

(i) Let $a + b\omega + c\omega^2 = x$ and $a + b\omega^2 + c\omega = y$. Then,

$$\begin{aligned} (a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 &= x^3 + y^3 \\ &= (x + y)(x + \omega y)(x + \omega^2 y) \end{aligned}$$

Now,

$$\begin{aligned} x + y &= (a + b\omega + c\omega^2) + (a + b\omega^2 + c\omega) \\ &= 2a + b(\omega + \omega^2) + c(\omega + \omega^2) \\ &= 2a - b - c \end{aligned}$$

$$\begin{aligned} x + \omega y &= (a + b\omega + c\omega^2) + \omega(a + b\omega^2 + c\omega) \\ &= (1 + \omega)a + (1 + \omega)b + 2\omega^2 c \\ &= \omega^2(2c - a - b) \end{aligned}$$

Similarly,

$$x + \omega^2 y = \omega(2b - a - c)$$

$$\begin{aligned} \Rightarrow (x + y)(x + \omega y)(x + \omega^2 y) &= \omega^3(2a - b - c)(2c - a - b)(2b - a - c) \\ &= (2a - b - c)(2c - a - b)(2b - a - c) \end{aligned}$$

(ii) $a + b + c = 0$

$$\Rightarrow b + c = -a, c + a = -b \text{ and } a + b = -c$$

Putting these values on the R.H.S. of result (i), we get

$$(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 = 27abc$$

ILLUSTRATION 3.31

Find the complex number ω satisfying the equation $z^3 = 8i$ and lying in the second quadrant on the complex plane.

Sol. $z^3 = 8i$

$$\text{or } z^3 = -8i^3$$

$$\text{or } \left(\frac{z}{-2i}\right)^3 = 1$$

$$\Rightarrow \frac{z}{-2i} = 1 \text{ or } \omega \text{ or } \omega^2$$

$$\Rightarrow z = -2i \text{ or } -2i\omega \text{ or } -2i\omega^2$$

$$\Rightarrow z = -2i \text{ or } -2i\left(\frac{-1 + \sqrt{3}i}{2}\right) \text{ or } -2i\left(\frac{-1 - \sqrt{3}i}{2}\right)$$

Hence, $z = -2i$ or $i + \sqrt{3}$ or $i - \sqrt{3}$ out of which $i - \sqrt{3}$ lies in the second quadrant.

ILLUSTRATION 3.32

If $\frac{1}{a+\omega} + \frac{1}{b+\omega} + \frac{1}{c+\omega} + \frac{1}{d+\omega} = \frac{1}{\omega}$, where, $a, b, c, d \in R$ and ω is a complex cube root of unity then find the value of $\sum \frac{1}{a^2 - a + 1}$.

Sol. $\frac{1}{a+\omega} + \frac{1}{b+\omega} + \frac{1}{c+\omega} + \frac{1}{d+\omega} = \frac{1}{\omega}$ (1)

Taking conjugate on both sides, we get

$$\frac{1}{a+\omega^2} + \frac{1}{b+\omega^2} + \frac{1}{c+\omega^2} + \frac{1}{d+\omega^2} = \frac{1}{\omega^2}$$
 (2)

Subtracting, we get

$$\sum \left(\frac{1}{a+\omega} - \frac{1}{a+\omega^2} \right) = \frac{1}{\omega} - \frac{1}{\omega^2}$$

$$\Rightarrow (\omega^2 - \omega) \sum \frac{1}{(a+\omega)(a+\omega^2)} = \omega^2 - \omega$$

$$\Rightarrow \sum \frac{1}{a^2 - a + 1} = 1$$

CONCEPT APPLICATION EXERCISE 3.4

1. If α and β are imaginary cube roots of unity, then find the value of $\alpha^4 + \beta^4 + 1/(\alpha\beta)$.
2. If ω is a complex cube root of unity, then find the value of $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \dots$ to $2n$ factors.
3. Write the complex number in $a + ib$ form using cube roots of unity:

(a) $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{1000}$ (b) If $z = \frac{(\sqrt{3} + i)^{17}}{(1 - i)^{50}}$

(c) $(i + \sqrt{3})^{100} + (i - \sqrt{3})^{100} + 2^{100}$

4. If $z + z^{-1} = 1$, then find the value of $z^{100} + z^{-100}$.
5. Find the common roots of $x^{12} - 1 = 0$, $x^4 + x^2 + 1 = 0$.
6. If α, β, γ are the roots of equation $x^3 - 3x^2 + 3x + 7 = 0$ and ω is cube roots of unity, then find the value of $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1}$.
7. Prove that $t^2 + 3t + 3$ is a factor of $(t+1)^{n+1} + (t+2)^{2n-1}$ for all integral values of $n \in N$.

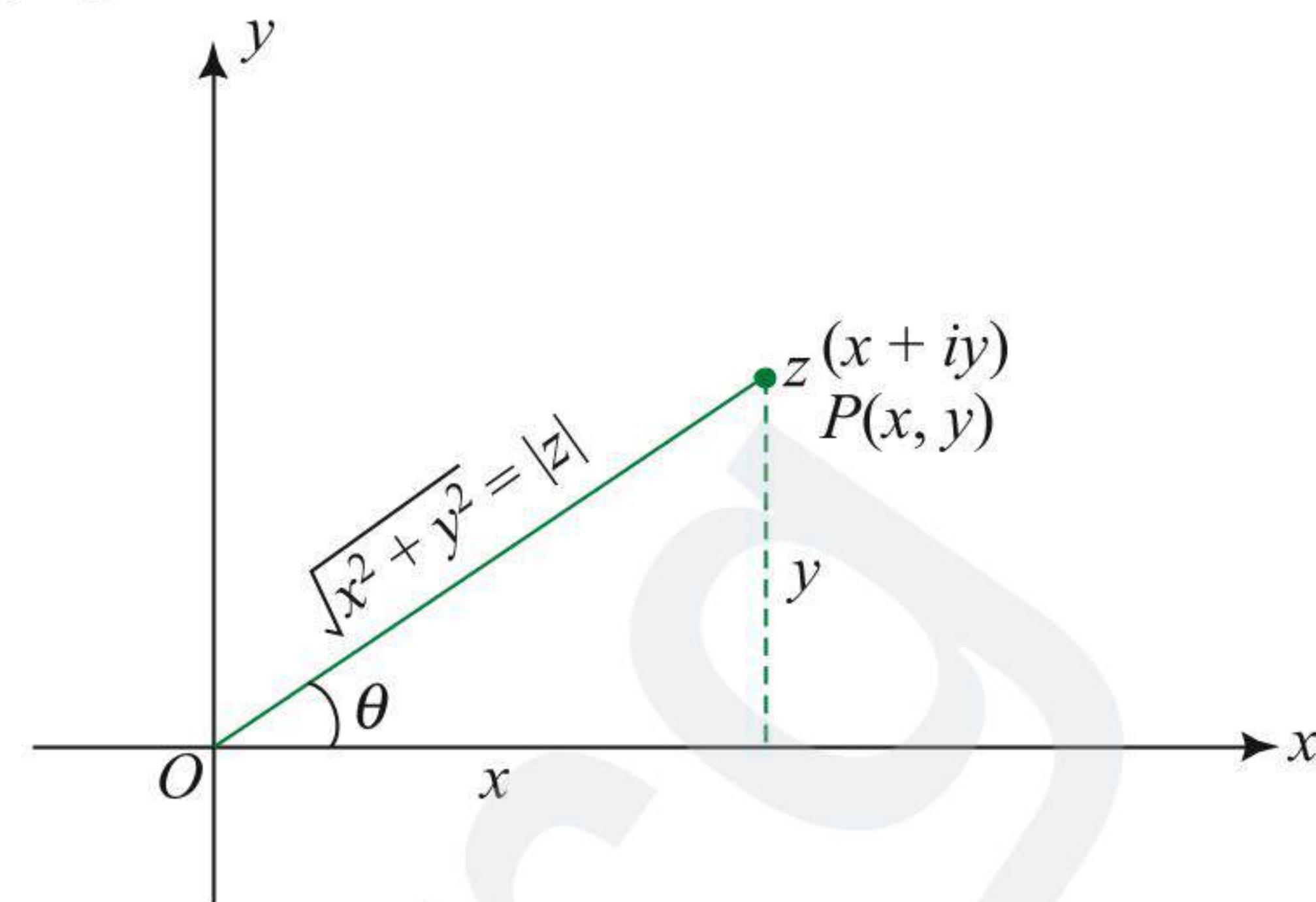
ANSWERS

1. 0
2. 1
3. (a) $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (b) $\frac{1}{2^8} \left(\frac{-1 - i\sqrt{3}}{2} \right)$ (c) 0
4. -1
5. $x = \pm\omega^2, \pm\omega$
6. $3\omega^2$

POLAR FORM OF COMPLEX NUMBER

We know that complex number $z = x + iy$ on the Argand plane represents the ordered pair (x, y) or point $P(x, y)$.

Let z lie in the first quadrant on Argand plane as shown in the following figure:



$$\begin{aligned} \text{Now, } z = x + iy &= \sqrt{x^2 + y^2} \left[\frac{x}{\sqrt{x^2 + y^2}} + i \frac{y}{\sqrt{x^2 + y^2}} \right] \\ &= |z|(\cos \theta + i \sin \theta) \end{aligned}$$

$z = |z|(\cos \theta + i \sin \theta)$ is called polar form of complex number.

It has mainly two components; modulus and argument.

MODULUS OF COMPLEX NUMBER

From the figure, $|z| = \sqrt{x^2 + y^2} = OP =$ distance of $(x + iy)$ from origin. $|z|$ is called modulus of complex number.

Clearly, $|z| = \sqrt{(\operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2}$ and $|z| \geq 0$.

For example,

(i) $|-3 + 4i| = \sqrt{(-3)^2 + (4)^2} = \sqrt{25} = 5$

(ii) $|-5 - \sqrt{-144}| = |-5 - 12i| = \sqrt{(-5)^2 + (-12)^2} = \sqrt{169} = 13$ etc.

Note

- We know that for two complex numbers z_1 and z_2 , $z_1 > z_2$ is meaningless; but $|z_1| > |z_2|$ means that the distance of z_1 from origin is more than the distance of z_2 from origin.
- Infinite complex numbers having same value of $|z|$ lie on circle which has centre at origin and radius as $|z|$.
- $|z| = |-z| = |\bar{z}| = |-\bar{z}|$

ARGUMENT OF COMPLEX NUMBER

The angle θ which OP makes with positive x -axis is called the argument or amplitude of z and is denoted by $\arg(z)$ or $\operatorname{amp}(z)$.

From the figure, we have

$$\tan \theta = \frac{y}{x} = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \Rightarrow \theta = \tan^{-1} \left(\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \right) = \arg(z)$$

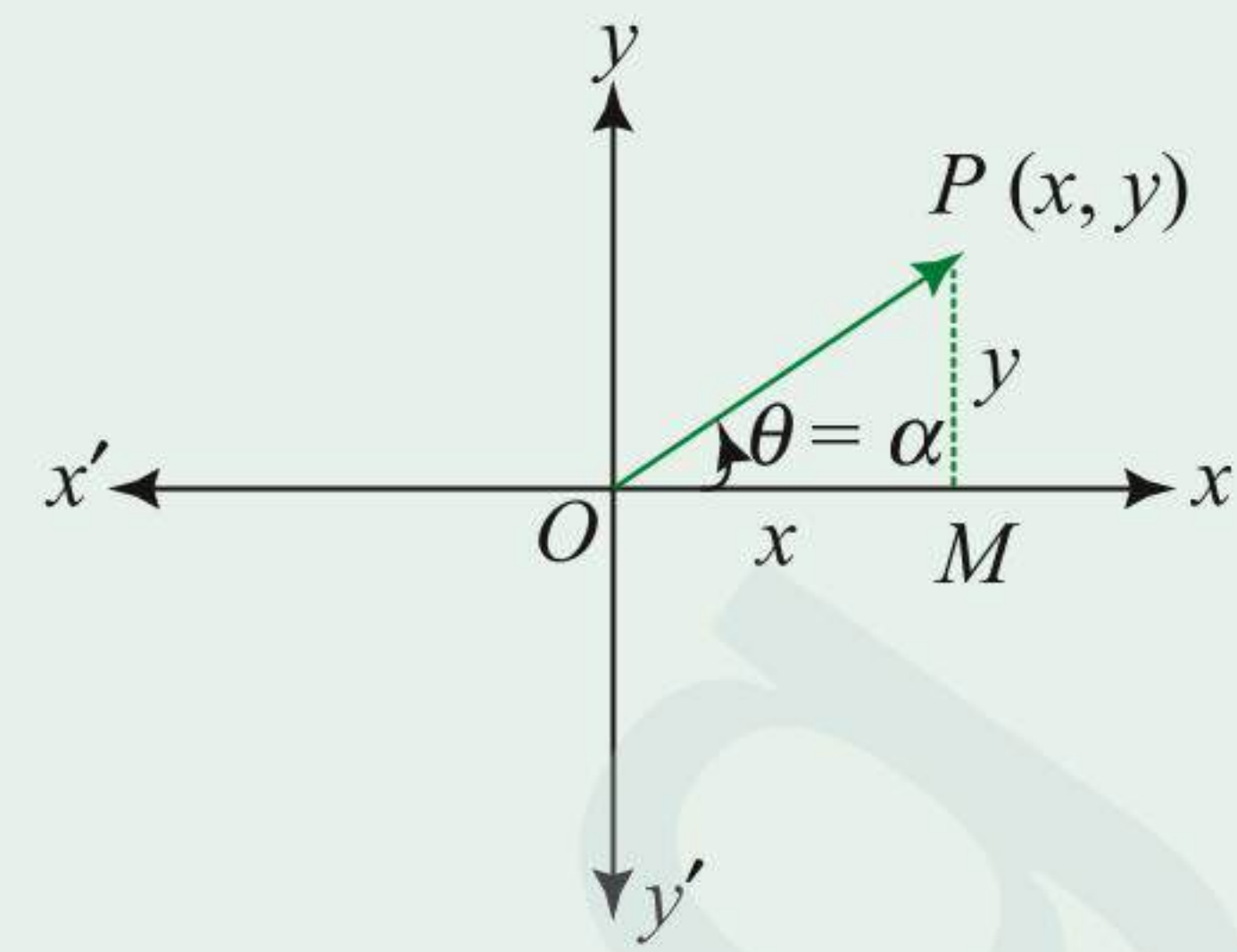
This angle θ has infinitely many values differing by multiples of 2π .

The unique value of θ such that $-\pi < \theta \leq \pi$ is called the principal value of the amplitude or principal argument. This formula for determining the argument of $z = x + iy$ has severe drawback, because $z_1 = 1 + i\sqrt{3}$ and $z_2 = -1 - i\sqrt{3}$ are two distinct complex numbers represented by two distinct points in the Argand plane, but their arguments seem to be $\tan^{-1}\sqrt{3} = \pi/3$ or $4\pi/3$ which is not correct. In fact, the argument is the common solution of the simultaneous trigonometric equations

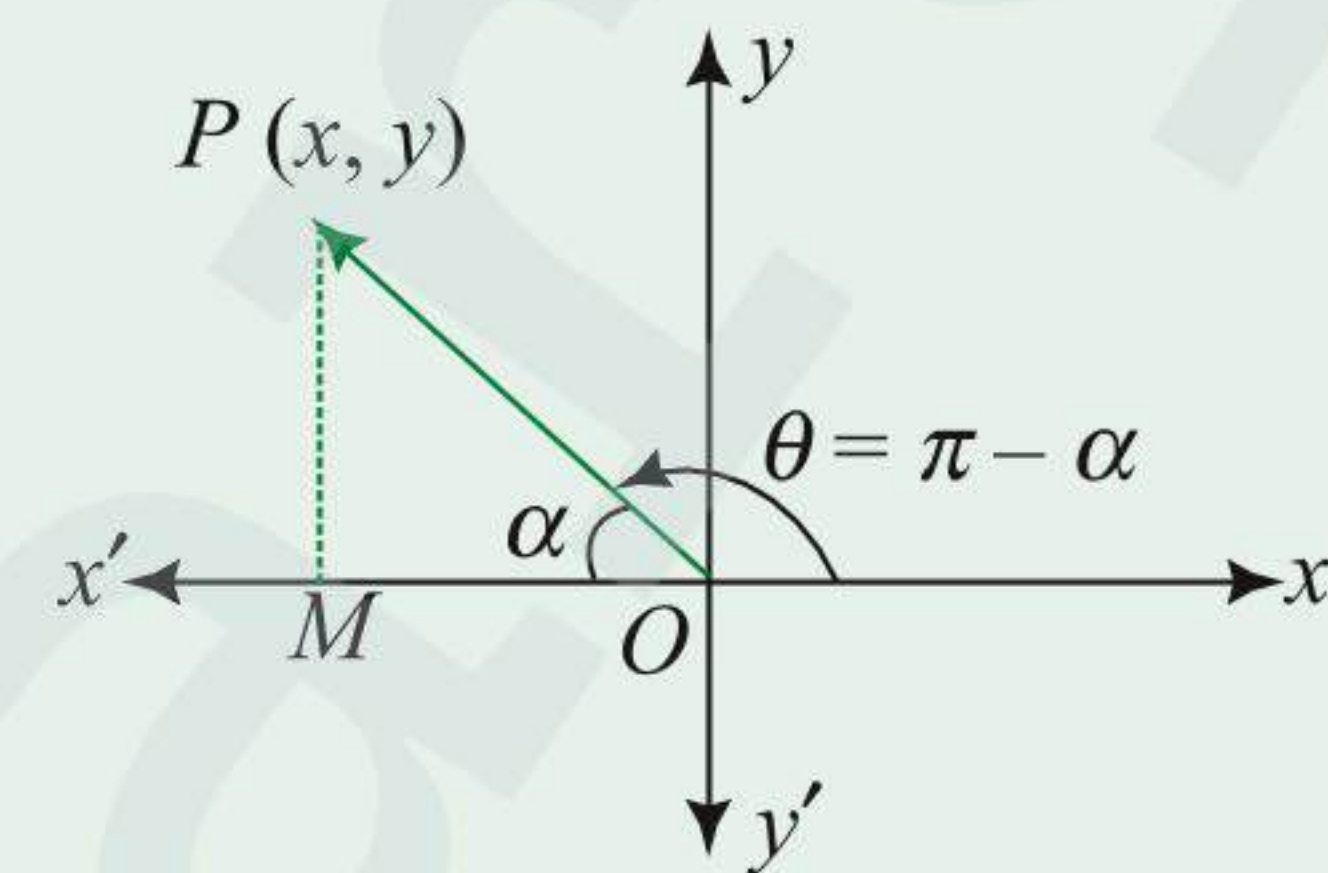
$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}} \text{ and } \sin \theta = \frac{y}{\sqrt{x^2 + y^2}} \text{ for } \theta \in (-\pi, \pi]$$

Argument of z in Different Quadrants

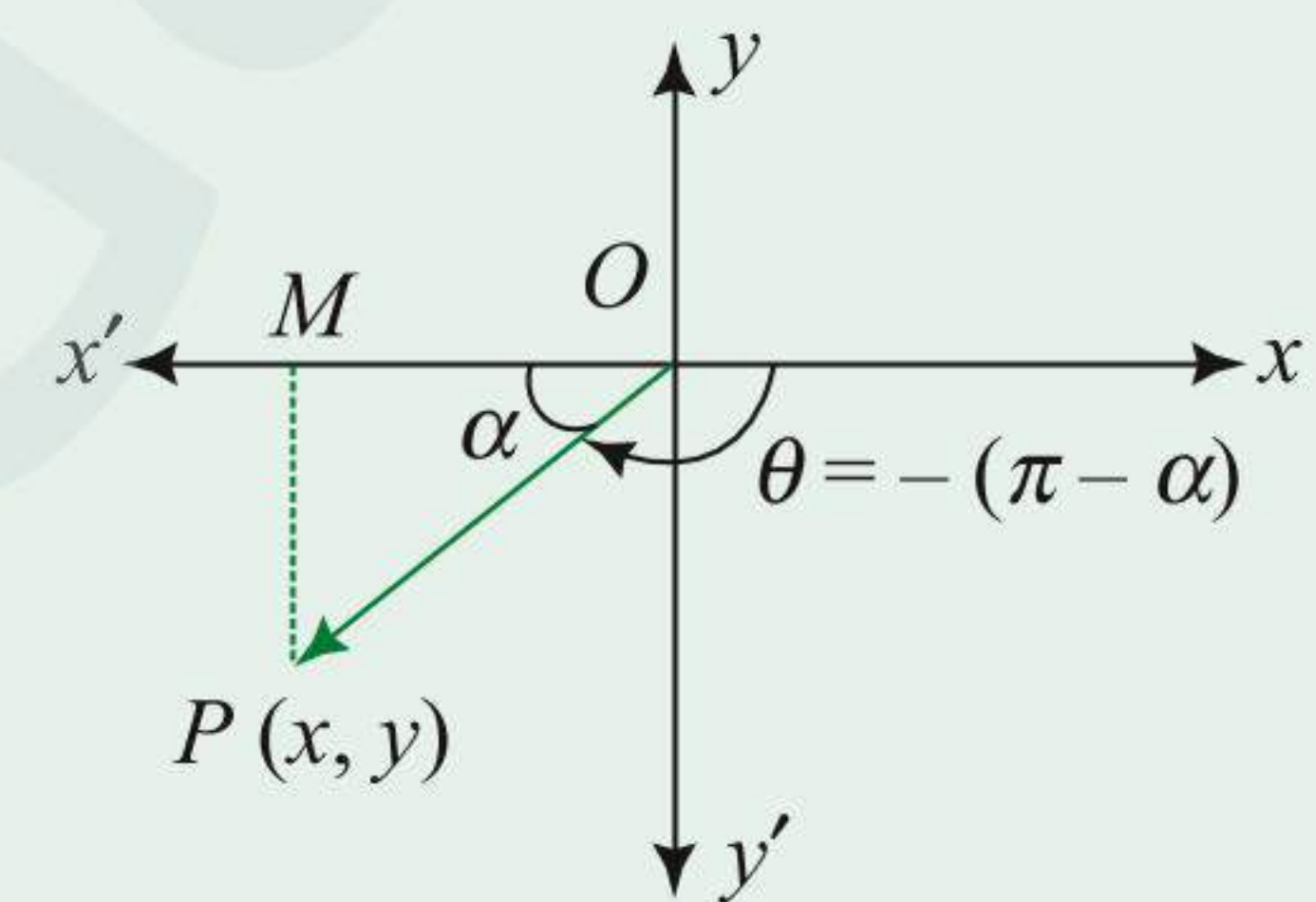
- (i) $z = x + iy$, z lies in the first quadrant ($x > 0$ and $y > 0$)
 From figure, $\tan \alpha = |y/x|$ and $\theta = \alpha$.
 Then $\arg(z) = \theta = \tan^{-1} |y/x|$.



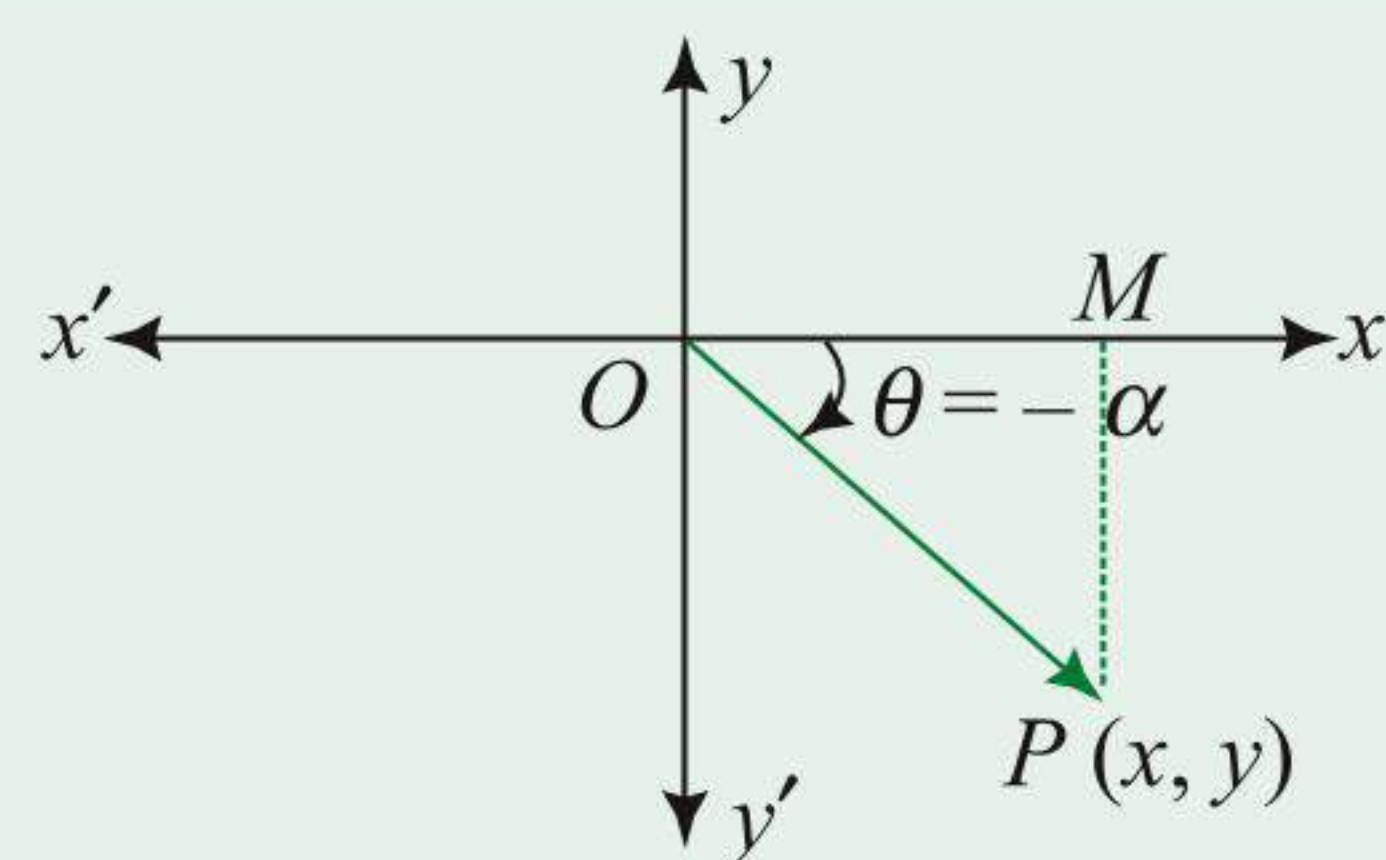
- (ii) $z = x + iy$, z lies in the second quadrant ($x < 0$ and $y > 0$)
 From figure, $\tan \alpha = |y/x|$ and $\theta = \pi - \alpha$.
 Then $\arg(z) = \theta = \pi - \alpha$, where α is the acute angle given by $\tan^{-1} |y/x|$.



- (iii) $z = x + iy$, z lies in the third quadrant ($x < 0$ and $y < 0$)
 From figure, $\tan \alpha = |y/x|$ and $\theta = -(\pi - \alpha) = -\pi + \alpha$.
 Then, $\arg(z) = \theta = \alpha - \pi$, where α is the acute angle given by $\tan \alpha = |y/x|$.

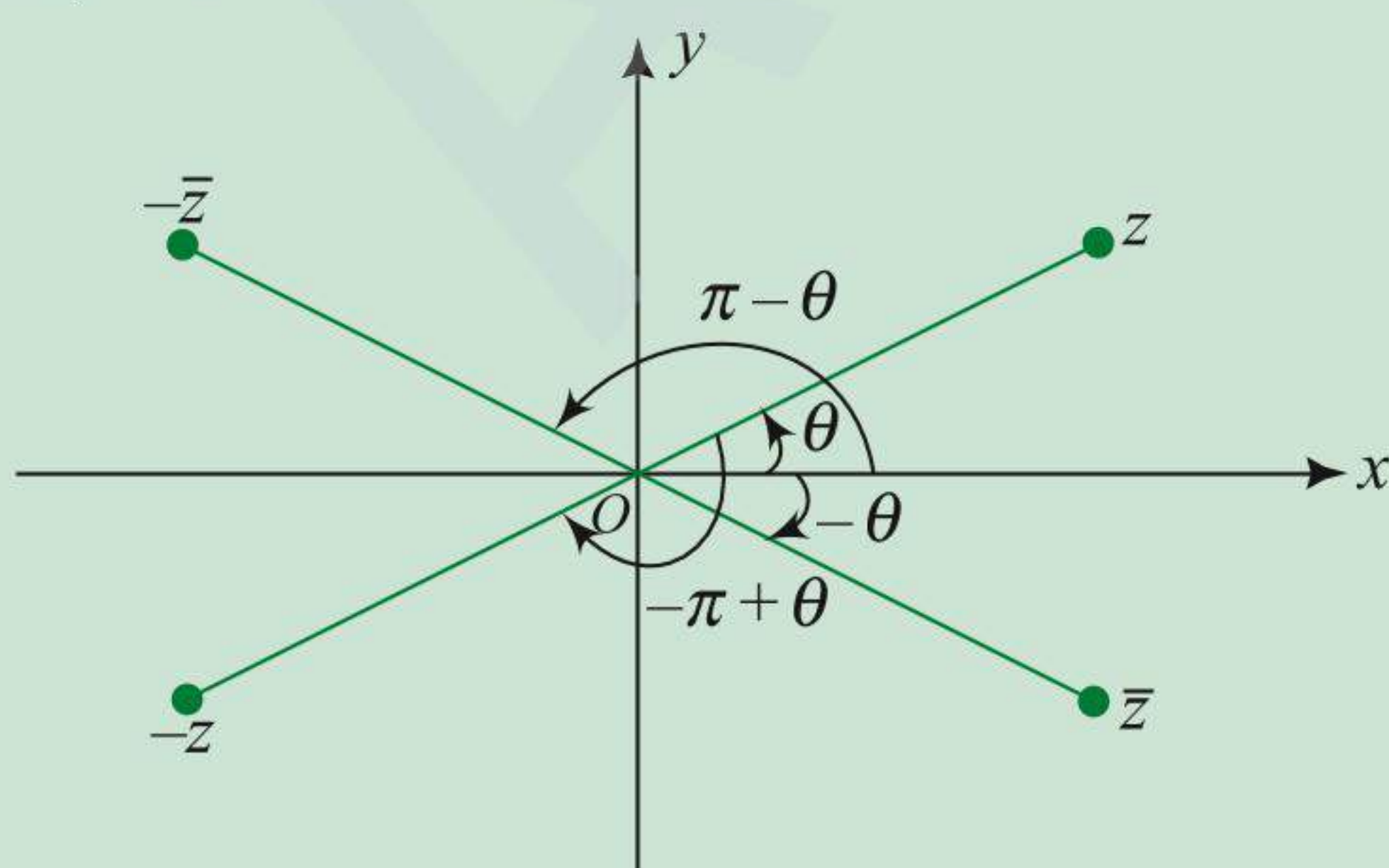


- (iv) $z = x + iy$, z lies in the fourth quadrant ($x > 0$ and $y < 0$)
 From figure, $\tan \alpha = |y/x|$ and $\theta = -\alpha$.
 Then $\arg(z) = \theta = -\alpha$, where α is the acute angle given by $\tan \alpha = |y/x|$.



Note

- $\arg(0 + i0)$ is not defined.
- Infinite complex numbers having same argument lie on the ray emanating from origin but excluding origin.
- If z lies on positive real axis $\arg(z) = 0$.
 If z lies on negative real axis, $\arg(z) = \pi$.
 If z lies on positive imaginary axis, $\arg(z) = \pi/2$.
 If z lies on negative imaginary axis, $\arg(z) = -\pi/2$.
- If $\arg(z) = \theta$ then $\arg(\bar{z}) = -\theta$, $\arg(-z) = (\theta - \pi)$ and $\arg(-\bar{z}) = \pi - \theta$.



- If two complex numbers z_1 and z_2 are same then $|z_1| = |z_2|$ and $\arg(z_1) = \arg(z_2)$.

ILLUSTRATION 3.33

Write the following complex numbers in polar form:

- (i) $-3\sqrt{2} + 3\sqrt{2}i$ (ii) $1 + i$ (iii) $-1 - i$
 (iv) $1 - i$ (v) $\frac{(1 + 7i)}{(2 - i)^2}$

Sol.

- (i) Let $z = -3\sqrt{2} + 3\sqrt{2}i$. Then,

$$|z| = \sqrt{(-3\sqrt{2})^2 + (3\sqrt{2})^2} = 6$$

$$\text{Let } \tan \alpha = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|} = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

Since the point representing z lies in the second quadrant, the argument of z is given by

$$\theta = \pi - \alpha = \pi - \left(\frac{\pi}{4}\right) = \left(\frac{3\pi}{4}\right)$$

So, the polar form of $z = -3\sqrt{2} + 3\sqrt{2}i$ is

$$z = |z| (\cos \theta + i \sin \theta) = 6 \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

(ii) Let $z = 1 + i$. Then $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Let

$$\tan \alpha = \left| \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \right|$$

$$\text{Then, } \tan \alpha = \left| \frac{1}{1} \right| = 1 \text{ or } \alpha = \frac{\pi}{4}$$

Since the point $(1, 1)$ representing z lies in the first quadrant, the argument of z is given by $\theta = \alpha = \pi/4$. So, the polar form of $z = 1 + i$ is

$$z = |z| (\cos \theta + i \sin \theta) = \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

(iii) Let $z = -1 - i$. Then $|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$.

$$\text{Let } \tan \alpha = \left| \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \right|.$$

$$\text{Then, } \tan \alpha = \left| \frac{-1}{-1} \right| = 1 \text{ or } \alpha = \frac{\pi}{4}$$

Since the point $(-1, -1)$ representing z lies in the third quadrant, the argument of z is given by

$$\theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{4}\right) = \frac{-3\pi}{4}$$

So, the polar form of $z = -1 - i$ is

$$z = |z| (\cos \theta + i \sin \theta) = \sqrt{2} \left\{ \cos \left(\frac{-3\pi}{4} \right) + i \sin \left(\frac{-3\pi}{4} \right) \right\}$$

(iv) Let $z = 1 - i$. Then, $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Let

$$\tan \alpha = \left| \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \right|.$$

$$\text{Then, } \tan \alpha = \left| \frac{-1}{1} \right| = 1 \text{ or } \alpha = \frac{\pi}{4}$$

Since the point $(1, -1)$ lies in the fourth quadrant, the argument of z is given by $\theta = -\alpha = -\pi/4$. So, the polar form of $z = 1 - i$ is

$$\begin{aligned} z &= |z| (\cos \theta + i \sin \theta) = \sqrt{2} \left\{ \cos \left(\frac{-\pi}{4} \right) + i \sin \left(\frac{-\pi}{4} \right) \right\} \\ &= \sqrt{2} \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right) \end{aligned}$$

(v) Let $z = (1 + 7i) / [(2 - i)^2]$. Then,

$$\begin{aligned} z &= \frac{1+7i}{4-4i+i^2} = \frac{1+7i}{3-4i} = \left(\frac{1+7i}{3-4i} \right) \left(\frac{3+4i}{3+4i} \right) \\ &= \frac{-25+25i}{25} = -1+i \end{aligned}$$

$$\therefore |z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

Let α be the acute angle given by

$$\tan \alpha = \left| \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \right| = \left| -\frac{1}{1} \right| = 1$$

Then $\alpha = \pi/4$. Since the point $(-1, 1)$ representing z lies in the second quadrant, we have $\theta = \arg(z) = \pi - \alpha = \pi - \pi/4 = 3\pi/4$. Hence, z in the polar form is given by

$$z = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

ILLUSTRATION 3.34

Let $z_1 = \cos 12^\circ + i \sin 12^\circ$ and $z_2 = \cos 48^\circ + i \sin 48^\circ$. Write complex number $(z_1 + z_2)$ in polar form. Find its modulus and argument.

$$\begin{aligned} \text{Sol. } z_1 + z_2 &= \cos 12^\circ + \cos 48^\circ + i (\sin 48^\circ + \sin 12^\circ) \\ &= 2 \cos 30^\circ \cdot \cos 18^\circ + 2i \sin 30^\circ \cdot \cos 18^\circ \\ &= 2 \cos 18^\circ (\cos 30^\circ + i \sin 30^\circ) \end{aligned}$$

$$\therefore |z| = 2 \cos 18^\circ$$

$$\text{and } \arg(z) = 30^\circ$$

ILLUSTRATION 3.35

Convert the complex number $z = 1 + \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}$ in polar form. Find its modulus and argument.

$$\begin{aligned} \text{Sol. } z &= 1 + \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5} \\ &= 2 \cos^2 \frac{4\pi}{5} + i 2 \sin \frac{4\pi}{5} \cos \frac{4\pi}{5} \\ &= 2 \cos \frac{4\pi}{5} \left(\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5} \right) \\ &= -2 \cos \frac{\pi}{5} \left(-\cos \frac{\pi}{5} + i \sin \frac{\pi}{5} \right) \\ &= 2 \cos \frac{\pi}{5} \left(\cos \frac{\pi}{5} - i \sin \frac{\pi}{5} \right) \\ &= 2 \cos \frac{\pi}{5} \left(\cos \left(-\frac{\pi}{5} \right) + i \sin \left(-\frac{\pi}{5} \right) \right) \end{aligned}$$

$$\text{Thus, } |z| = 2 \cos \frac{\pi}{5} \text{ and } \arg(z) = -\frac{\pi}{5}.$$

ILLUSTRATION 3.36

Let z and w be two nonzero complex numbers such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$. Then prove that $z = -\bar{w}$.

Sol. Let

$\arg(w) = \theta$. Then $\arg(z) = \pi - \theta$. Therefore,

$$w = |w| (\cos \theta + i \sin \theta)$$

$$\text{and } z = |z| (\cos (\pi - \theta) + i \sin (\pi - \theta))$$

$$= |w| (-\cos \theta + i \sin \theta)$$

$$= -|w| (\cos \theta - i \sin \theta) = -\bar{w}.$$

$$[\because |z| = |w|]$$

ILLUSTRATION 3.37

Find nonzero integral solutions of $|1 - i|^x = 2^x$.

Sol. We have,

$$|1 - i|^x = 2^x$$

$$\left(\sqrt{1^2 + (-1)^2}\right)^x = 2^x$$

$$\Rightarrow (\sqrt{2})^x = 2^x$$

$$\text{or } 2^{x/2} = 2^x$$

$$\text{or } 2^{x/2} = 1$$

$$\text{or } 2^{x/2} = 2^0 \quad \text{or } \frac{x}{2} = 0 \quad \text{or } x = 0$$

Hence, the given equation has no solution.

ILLUSTRATION 3.38

Let z be a complex number satisfying $|z| = 3|z - 1|$. Then

prove that $\left|z - \frac{9}{8}\right| = \frac{3}{8}$.

Sol. Let $z = x + iy$.

$$\text{Now, } |z| = 3|z - 1|$$

$$\Rightarrow |z|^2 = 9|z - 1|^2$$

$$\Rightarrow |x + iy|^2 = 9|(x - 1) + iy|^2$$

$$\Rightarrow x^2 + y^2 = 9[(x - 1)^2 + y^2]$$

$$\Rightarrow 8x^2 + 8y^2 - 18x + 9 = 0$$

$$\Rightarrow \left(x - \frac{9}{8}\right)^2 + y^2 = \frac{9}{64}$$

$$\Rightarrow \left|\left(x - \frac{9}{8}\right) + iy\right| = \frac{3}{8}$$

$$\Rightarrow \left|z - \frac{9}{8}\right| = \frac{3}{8}$$

ILLUSTRATION 3.39

If complex number $z = x + iy$ satisfies the equation $\text{Re}(z + 1) = |z - 1|$, then prove that z lies on $y^2 = 4x$.

Sol. We have $\text{Re}(z + 1) = |z - 1|$

$$\Rightarrow \text{Re}(x + iy + 1) = |x + iy - 1|$$

$$\Rightarrow x + 1 = \sqrt{(x - 1)^2 + y^2}$$

$$\Rightarrow (x + 1)^2 = (x - 1)^2 + y^2$$

$$\Rightarrow y^2 = 4x$$

Thus, z lies on $y^2 = 4x$.

ILLUSTRATION 3.40

Solve the equation $|z| = z + 1 + 2i$.

Sol. $|z| = z + 1 + 2i$

$$\Rightarrow \sqrt{x^2 + y^2} = x + iy + 1 + 2i = x + 1 + (2 + y)i$$

$$\Rightarrow \sqrt{x^2 + y^2} = x + 1 \text{ and } 0 = 2 + y \text{ or } y = -2$$

$$\Rightarrow \sqrt{x^2 + 4} = x + 1$$

$$\text{or } x^2 + 4 = x^2 + 2x + 1$$

$$\text{or } 2x = 3$$

$$\text{or } x = \frac{3}{2}$$

$$\Rightarrow x + iy = \frac{3}{2} - 2i$$

ILLUSTRATION 3.41

Find the range of real number α for which the equation $z + \alpha|z - 1| + 2i = 0$ has a solution.

Sol. Let $z = x + iy$.

We have,

$$z + \alpha|z - 1| + 2i = 0$$

$$\Rightarrow x + i(y + 2) + \alpha\sqrt{(x - 1)^2 + y^2} = 0$$

Equating real and imaginary parts

$$\therefore y = -2 \text{ and } x + \alpha\sqrt{(x - 1)^2 + 4} = 0$$

$$\therefore x^2 = \alpha^2(x^2 - 2x + 5)$$

$$\therefore (1 - \alpha^2)x^2 + 2\alpha^2x - 5\alpha^2 = 0$$

Since x is real,

$$\therefore D = b^2 - 4ac \geq 0$$

$$\Rightarrow 4\alpha^4 + 20\alpha^2(1 - \alpha^2) \geq 0$$

$$\Rightarrow -4\alpha^4 + 5\alpha^2 \geq 0$$

$$\Rightarrow 4\alpha^2\left(\alpha^2 - \frac{5}{4}\right) \leq 0$$

$$\Rightarrow \frac{-\sqrt{5}}{2} \leq \alpha \leq \frac{\sqrt{5}}{2}.$$

ILLUSTRATION 3.42

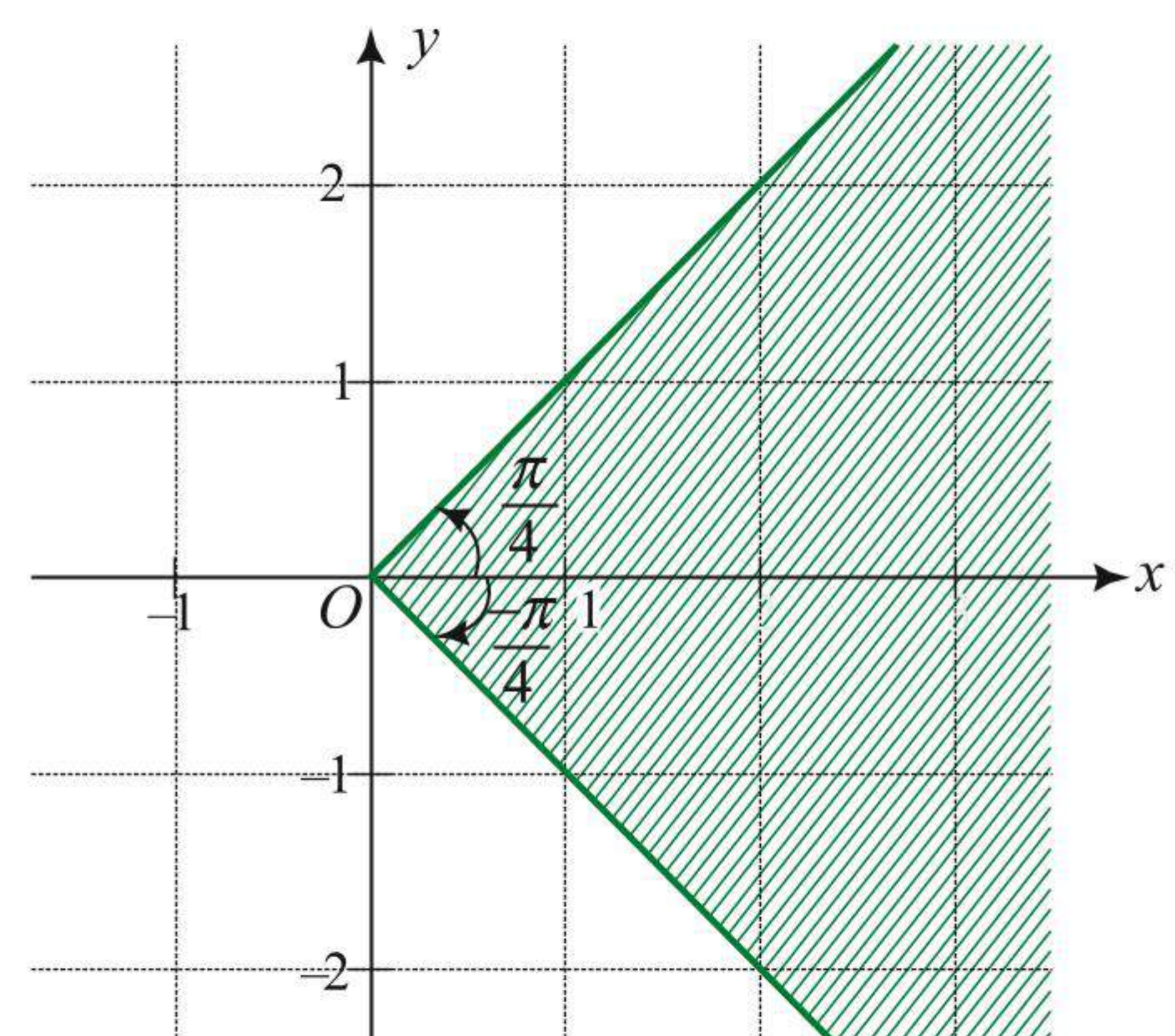
Find the area of the region bounded by $|\arg z| \leq \pi/4$ and $|z - 1| < |z - 3|$.

Sol. $|\arg z| < \pi/4$

$$-\pi/4 < \arg z < \pi/4$$

(1)

which represents the region shown in the figure.



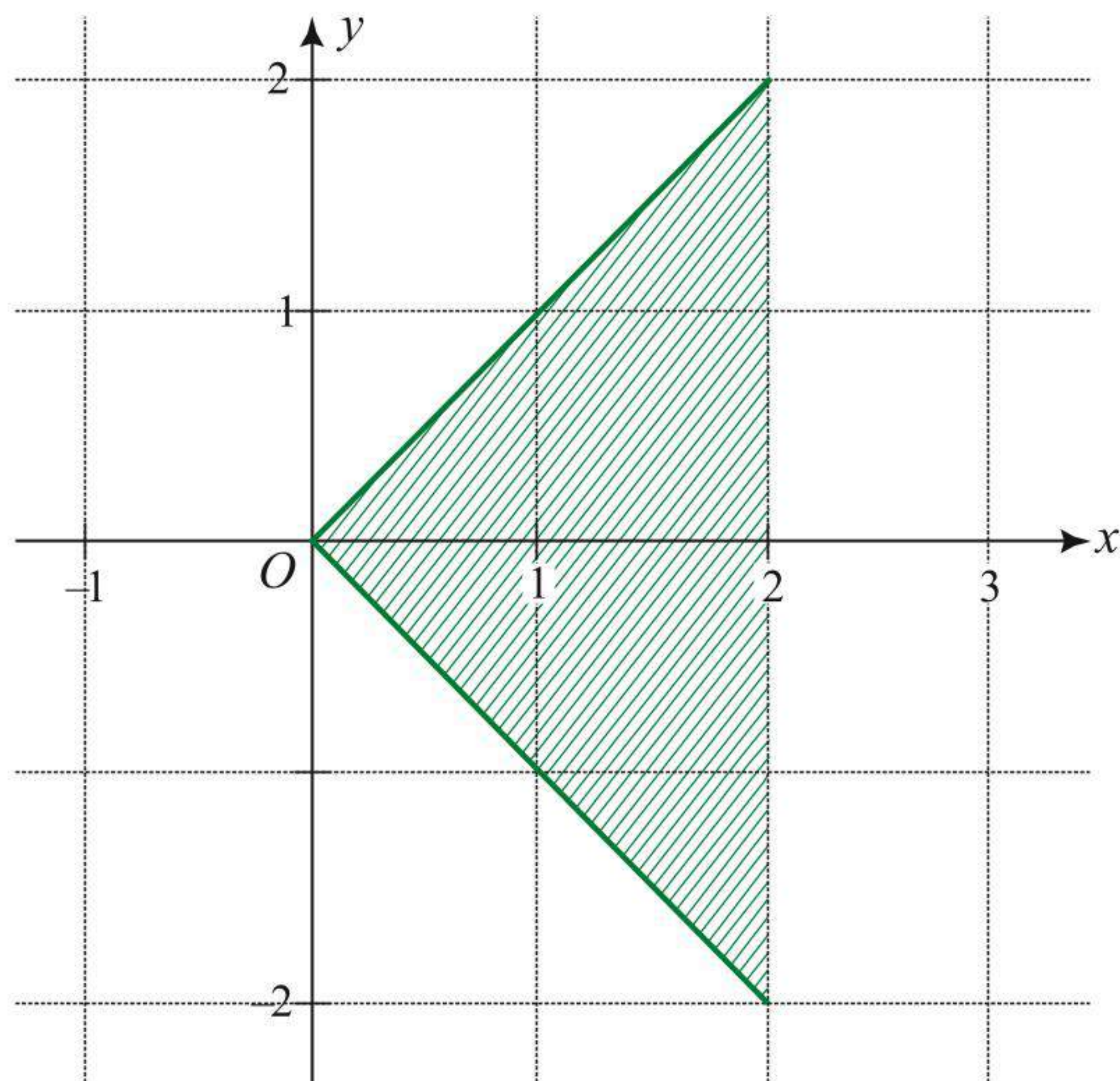
$$|z - 1| < |z - 3|$$

$$\Rightarrow (x - 1)^2 + y^2 < (x - 3)^2 + y^2$$

$$\Rightarrow x < 2$$

(2)

Common region of (1) and (2) is shown in the figure.



Area of the shaded region is $\frac{1}{2}(4)(2) = 4$ sq. units.

ILLUSTRATION 3.43

Prove that triangle formed by complex numbers z_1, z_2 and z_3 is equilateral if $|z_1| = |z_2| = |z_3|$ and $z_1 + z_2 + z_3 = 0$.

Sol. Triangle is formed by complex numbers z_1, z_2 and z_3 .

Let $|z_1| = |z_2| = |z_3|$ and $z_1 + z_2 + z_3 = 0$.

If $|z_1| = |z_2| = |z_3|$, then z_1, z_2 and z_3 are equidistant from origin. So, origin is circumcentre of the triangle.

Also, centroid of the triangle is $\frac{z_1 + z_2 + z_3}{3} = 0$.

Thus, circumcentre and centroid coincide.

Hence, triangle is equilateral.

EULER'S FORM OF COMPLEX NUMBER

For complex number $z = \cos \theta + i \sin \theta$, Euler's form is $e^{i\theta} = \cos \theta + i \sin \theta$.

Here, symbol e is known as the exponential constant which is an irrational number having approximate value 2.718. It is also known as Euler's number and Napier's constant. The value of e

arises as the limit of $\left(1 + \frac{1}{n}\right)^n$ when n approaches infinity.

We have the expansion formulas for e^x , $\sin x$ and $\cos x$:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

$$\begin{aligned} \text{Now, } e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) \\ &= \cos \theta + i \sin \theta \end{aligned}$$

More generally, for any complex number $z = x + iy$, Euler's form is given by $z = |z|(\cos \theta + i \sin \theta) = |z| e^{i\theta}$

With the use of this form, study of complex numbers and their properties becomes easier.

ILLUSTRATION 3.44

Show that $e^{2mi\theta} \left(\frac{i \cot \theta + 1}{i \cot \theta - 1} \right)^m = 1$.

Sol. Let $\cot^{-1} p = \theta$. Then $\cot \theta = p$. Now,

$$\begin{aligned} \text{L.H.S.} &= e^{2mi\theta} \left(\frac{i \cot \theta + 1}{i \cot \theta - 1} \right)^m \\ &= e^{2mi\theta} \left[\frac{i(\cot \theta - i)}{i(\cot \theta + i)} \right]^m \\ &= e^{2mi\theta} \left(\frac{\cot \theta - i}{\cot \theta + i} \right)^m \\ &= e^{2mi\theta} \left(\frac{\cos \theta - i \sin \theta}{\cos \theta + i \sin \theta} \right)^m \\ &= e^{2mi\theta} \left(\frac{e^{-i\theta}}{e^{i\theta}} \right)^m \\ &= e^{2mi\theta} (e^{-2i\theta})^m \\ &= e^{2mi\theta} (e^{-2i\theta})^m = e^0 = 1 = \text{R.H.S.} \end{aligned}$$

ILLUSTRATION 3.45

$Z_1 \neq Z_2$ are two points in an Argand plane. If $a|Z_1| = b|Z_2|$, then prove that $\frac{aZ_1 - bZ_2}{aZ_1 + bZ_2}$ is purely imaginary.

Sol. Let $Z_1 = r_1 e^{i\theta}$, $Z_2 = r_2 e^{i(\theta + \alpha)}$.

Given that $ar_1 = br_2$

$$\begin{aligned} \therefore Z &= \frac{aZ_1 - bZ_2}{aZ_1 + bZ_2} = \frac{e^{i\theta} - e^{i(\theta + \alpha)}}{e^{i\theta} + e^{i(\theta + \alpha)}} \\ &= \frac{1 - e^{i\alpha}}{1 + e^{i\alpha}} \quad (\text{Dividing Nr. and Dr. by } e^{i\theta}) \\ &= \frac{e^{-i\alpha/2} - e^{i\alpha/2}}{e^{-i\alpha/2} + e^{i\alpha/2}} \quad (\text{Dividing Nr. and Dr. by } e^{i\alpha/2}) \\ &= \frac{-2i \sin \frac{\alpha}{2}}{2 \cos \frac{\alpha}{2}} \\ &= -i \tan \frac{\alpha}{2} \end{aligned}$$

Hence, Z is purely imaginary.

LOGARITHM OF COMPLEX NUMBER

Euler's form of a complex number helps to find its logarithm. For any complex number $z = x + iy$, Euler form is $(|z|e^{i\theta})$.

$$\begin{aligned}
 \therefore \log_e(x + iy) &= \log_e(|z|e^{i\theta}) \\
 &= \log_e|z| + \log_e e^{i\theta} \\
 &= \log_e|z| + i\theta \\
 &= \log_e \sqrt{x^2 + y^2} + i \arg(z)
 \end{aligned}$$

ILLUSTRATION 3.46Find the real part of $(1 - i)^{-i}$.**Sol.** Let $z = (1 - i)^{-i}$. Taking log on both sides, we have

$$\begin{aligned}
 \log z &= -i \log_e (1 - i) \\
 &= -i \log_e \left(\sqrt{2} \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right) \right) \\
 &= -i \log_e (\sqrt{2} e^{-i(\pi/4)}) \\
 &= -i \left[\frac{1}{2} \log_e 2 + \log_e e^{-i\pi/4} \right] \\
 &= -i \left[\frac{1}{2} \log_e 2 - \frac{i\pi}{4} \right] \\
 &= -\frac{i}{2} \log_e 2 - \frac{\pi}{4}
 \end{aligned}$$

$$\Rightarrow z = e^{-\pi/4} e^{-i(\log 2)/2}$$

$$\Rightarrow \operatorname{Re}(z) = e^{-\pi/4} \cos \left(\frac{1}{2} \log 2 \right)$$

CONCEPT APPLICATION EXERCISE 3.5

- Find the principal argument of each of the following:
 - $-1 - i\sqrt{3}$
 - $\frac{1 + \sqrt{3}i}{\sqrt{3} + i}$
 - $\sin \alpha + i(1 - \cos \alpha)$, $0 < \alpha < \pi$
 - $(1 + i\sqrt{3})^2$
- Find the modulus, argument and the principal argument of the complex number $(\tan 1 - i)^2$.
- If $\frac{3\pi}{2} < \alpha < 2\pi$, find the modulus and argument of $(1 - \cos 2\alpha) + i \sin 2\alpha$.
- Find the principal argument of the complex number $\sin \frac{6\pi}{5} + i \left(1 + \cos \frac{6\pi}{5} \right)$.
- If $z = re^{i\theta}$, then prove that $|e^{iz}| = e^{-r \sin \theta}$.
- Find the complex number z satisfying $\operatorname{Re}(z^2) = 0$, $|z| = \sqrt{3}$.
- If $|z - i \operatorname{Re}(z)| = |z - \operatorname{Im}(z)|$, then prove that z lies on the bisectors of the quadrants.
- Find the locus of the points representing the complex number z for which $|z + 5|^2 - |z - 5|^2 = 10$.
- Solve: $z^2 + |z| = 0$.
- Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by $A = \{z : |z| \leq 2\}$ and $B = \{z : (1 - i)z + (1 + i)\bar{z} \geq 4\}$. Find the area of region $A \cap B$.
- Find the real part of $e^{e^{i\theta}}$.
- Prove that $z = i^i$, where $i = \sqrt{-1}$, is purely real.

ANSWERS

- (a) $\frac{-2\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\alpha}{2}$ (d) $\frac{2\pi}{3}$
- Modulus = $\sec^2 1$; Argument = $2 - \pi$
- Modulus = $-2 \sin \alpha$; Argument = $\frac{3\pi}{2} - \alpha$
- $\frac{9\pi}{10}$
- $z = \pm \sqrt{\frac{3}{2}} \pm i \sqrt{\frac{3}{2}}$
- $x = \frac{1}{2}$
- $0, i, -i$
- $\pi - 2$
- $e^{\cos \theta} [\cos(\sin \theta)]$

MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS IN POLAR FORM

Consider two complex numbers,

$$z_1 = |z_1|(\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = |z_2|(\cos \theta_2 + i \sin \theta_2)$$

$$\begin{aligned}
 z_1 z_2 &= |z_1| |z_2| (\cos \theta_1 + i \sin \theta_1) (\cos \theta_2 + i \sin \theta_2) \\
 &= |z_1| |z_2| [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\
 &\quad + i(\sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1)] \\
 &= |z_1| |z_2| [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]
 \end{aligned}$$

Thus, $|z_1 z_2| = |z_1| |z_2|$ and $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

From these results, we have

$$|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$$

and $\arg(z_1 z_2 z_3 \dots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \dots + \arg(z_n)$ If $z_1 = z_2 = \dots = z_n = z$, then $|z^n| = |z|^n$ and $\arg(z^n) = n \arg(z)$.

$$\begin{aligned}
 \text{Now, } \frac{z_1}{z_2} &= \frac{|z_1|(\cos \theta_1 + i \sin \theta_1)}{|z_2|(\cos \theta_2 + i \sin \theta_2)} \\
 &= \frac{|z_1|(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{|z_2|(\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)} \\
 &\quad [(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + \\
 &= \frac{|z_1|}{|z_2|} \cdot \frac{i(\sin \theta_1 \cos \theta_2 - \sin \theta_2 \cos \theta_1)]}{\cos^2 \theta_2 + \sin^2 \theta_2} \\
 &= \frac{|z_1|}{|z_2|} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]
 \end{aligned}$$

$$\text{Thus, } \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

$$\text{and } \arg \left(\frac{z_1}{z_2} \right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$$

The above results can also be proved using Euler's formula.

$$z_1 = |z_1| e^{i\theta_1} \text{ and } z_2 = |z_2| e^{i\theta_2}$$

$$\therefore z_1 z_2 = |z_1| |z_2| e^{i\theta_1} e^{i\theta_2} = |z_1| |z_2| e^{i(\theta_1 + \theta_2)}$$

And $\frac{z_1}{z_2} = \frac{|z_1| e^{i\theta_1}}{|z_2| e^{i\theta_2}} = \frac{|z_1|}{|z_2|} e^{i(\theta_1 - \theta_2)}$

Using Euler's formula we have $|z^n| = |z|^n$ and $\arg(z^n) = n \arg(z)$ for $n \in \mathbb{Q}$.

ILLUSTRATION 3.47

If $(\sqrt{8} + i)^{50} = 3^{49} (a + ib)$, then find the value of $a^2 + b^2$.

Sol. Given that

$$(\sqrt{8} + i)^{50} = 3^{49} (a + ib)$$

Taking modulus and squaring both sides, we get

$$(8 + 1)^{50} = 3^{98} (a^2 + b^2)$$

$$\text{or } 9^{50} = 3^{98} (a^2 + b^2)$$

$$\text{or } 3^{100} = 3^{98} (a^2 + b^2)$$

$$\text{or } (a^2 + b^2) = 9$$

ILLUSTRATION 3.48

Show that $(x^2 + y^2)^4 = (x^4 - 6x^2y^2 + y^4)^2 + (4x^3y - 4xy^3)^2$.

Sol. $(x^2 + y^2)^4 = |x + iy|^8$

$$= |(x + iy)^2|^4$$

$$= |(x^2 - y^2) + 2ixy|^4$$

$$= [(x^2 - y^2) + 2ixy]^2$$

$$= |(x^2 - y^2)^2 + (2ixy)^2 + 2(x^2 - y^2)(2ixy)|^2$$

$$= |x^4 + y^4 - 2x^2y^2 - 4x^2y^2 + i(4x^3y - 4xy^3)|^2$$

$$= |x^4 + y^4 - 6x^2y^2 + i(4x^3y - 4xy^3)|^2$$

$$= (x^4 - 6x^2y^2 + y^4)^2 + (4x^3y - 4xy^3)^2$$

ILLUSTRATION 3.49

If $\arg(z_1) = 170^\circ$ and $\arg(z_2) = 70^\circ$, then find the principal argument of $z_1 z_2$.

Sol. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = 170^\circ + 70^\circ = 240^\circ$

Thus, $z_1 z_2$ lies in the third quadrant. Hence, its principal argument is -120° .

ILLUSTRATION 3.50

Find the value of the expression

$$\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \left(\cos \frac{\pi}{2^2} + i \sin \frac{\pi}{2^2} \right) \dots \text{to } \infty$$

Sol. $\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \left(\cos \frac{\pi}{2^2} + i \sin \frac{\pi}{2^2} \right) \dots \text{to } \infty$

$$= \cos \left(\frac{\pi}{2} + \frac{\pi}{2^2} + \dots \right) + i \sin \left(\frac{\pi}{2} + \frac{\pi}{2^2} + \dots \right)$$

$$= \cos \left[\frac{\pi}{2} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots \right) \right] + i \sin \left[\frac{\pi}{2} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots \right) \right]$$

$$= \cos \left[\frac{\pi}{2} \left(\frac{1}{1 - \frac{1}{2}} \right) \right] + i \sin \left[\frac{\pi}{2} \left(\frac{1}{1 - \frac{1}{2}} \right) \right] = \cos \pi + i \sin \pi = -1$$

ILLUSTRATION 3.51

Find the principal argument of the complex number

$$\frac{(1+i)^5 (1+\sqrt{3}i)^2}{-2i(-\sqrt{3}+i)}$$

Sol. $Z = \frac{(1+i)^5 (1+\sqrt{3}i)^2}{-2i(-\sqrt{3}+i)}$

$$= \frac{(\sqrt{2})^5 \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right)^5 \cdot 2^2 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right)^2}{(2i)2 \left(\frac{\sqrt{3}}{2} - \frac{i}{2} \right)}$$

$$\therefore \arg(Z) = 5 \arg \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) + 2 \arg \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) - \arg(i) - \arg \left(\frac{\sqrt{3}}{2} - \frac{i}{2} \right)$$

$$(\because \arg(\sqrt{2})^5, \arg(2^2), \arg(4) = 0)$$

$$= \frac{5\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{2} + \frac{\pi}{6} = \frac{19\pi}{12} = 2\pi - \frac{5\pi}{12}$$

Therefore, Z lies in the fourth quadrant.

Hence, principal argument is $-\frac{5\pi}{12}$

ILLUSTRATION 3.52

If $z = \frac{(\sqrt{3} + i)^{17}}{(1 - i)^{50}}$, then find $\arg(z)$.

Sol. $z = \frac{(\sqrt{3} + i)^{17}}{(1 - i)^{50}} = \frac{\left(2 \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right) \right)^{17}}{\left(\sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) \right)^{50}}$

$$\Rightarrow \arg(z) = 17 \arg \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right) - 50 \arg \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right)$$

$$= 17 \left(\frac{\pi}{6} \right) - 50 \left(-\frac{\pi}{4} \right)$$

$$= \frac{17\pi}{6} + \frac{25\pi}{2} = \frac{46\pi}{3} = 15\pi + \frac{\pi}{3}$$

Thus, z lies in third quadrant with principal argument $-\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$

ILLUSTRATION 3.53

If $z = x + iy$ and $w = (1 - iz)/(z - i)$, then show that $|w| = 1 \Rightarrow z$ is purely real.

Sol. We have,

$$|w| = 1$$

$$\Rightarrow \left| \frac{1 - iz}{z - i} \right| = 1 \quad \text{or} \quad \frac{|1 - iz|}{|z - i|} = 1$$

$$\begin{aligned}
&\text{or } |1 - iz| = |z - i| \quad (1) \\
&\text{or } |1 - i(x + iy)| = |x + iy - i|, \text{ where } z = x + iy \\
&\text{or } |1 + y - ix| = |x + i(y - 1)| \\
&\text{or } \sqrt{(1+y)^2 + (-x)^2} = \sqrt{x^2 + (y-1)^2} \\
&\text{or } (1+y)^2 + x^2 = x^2 + (y-1)^2 \\
&\text{or } y = 0 \\
&\Rightarrow z = x + i0 = x, \text{ which is purely real}
\end{aligned}$$

ILLUSTRATION 3.54

It is given that complex numbers z_1 and z_2 satisfy $|z_1| = 2$ and $|z_2| = 3$. If the included angle of their corresponding vectors is

60° , then find the value of $\left| \frac{z_1 + z_2}{z_1 - z_2} \right|$.

Sol. Let $z_1 = 2(\cos \theta + i \sin \theta)$

$$\Rightarrow z_2 = 3((\cos(\theta + 60^\circ) + i \sin(\theta + 60^\circ)))$$

$$\Rightarrow \frac{z_2}{z_1} = \frac{3(\cos(\theta + 60^\circ) + i \sin(\theta + 60^\circ))}{2(\cos \theta + i \sin \theta)}$$

$$\text{or } \frac{z_2}{z_1} = \frac{3}{2}(\cos 60^\circ + i \sin 60^\circ) = \frac{3}{2}\left(\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)$$

$$\begin{aligned}
\therefore \left| \frac{z_1 + z_2}{z_1 - z_2} \right| &= \left| \frac{1 + \frac{z_2}{z_1}}{1 - \frac{z_2}{z_1}} \right| = \left| \frac{1 + \frac{3}{2}\left(\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)}{1 - \frac{3}{2}\left(\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)} \right| = \left| \frac{7 + i3\sqrt{3}}{1 - i3\sqrt{3}} \right| \\
&= \frac{\sqrt{49 + 27}}{\sqrt{1 + 27}} = \frac{\sqrt{76}}{\sqrt{28}} = \frac{\sqrt{19}}{\sqrt{7}}
\end{aligned}$$

ILLUSTRATION 3.55

Solve the equation $z^3 = \bar{z}$ ($z \neq 0$).

Sol. As $z \neq 0$,

$$\begin{aligned}
&|z^3| = |\bar{z}| \\
\Rightarrow &|z|^3 = |\bar{z}| \\
\Rightarrow &|z|^3 = |z| \\
\Rightarrow &|z| = 1 \\
\therefore &z^3 = \bar{z} \\
\Rightarrow &z^4 = z\bar{z} = |z|^2 = 1 \\
\Rightarrow &z = \pm 1, \pm i
\end{aligned}$$

ILLUSTRATION 3.56

If $2z_1/3z_2$ is a purely imaginary number, then find the value of $|(z_1 - z_2)/(z_1 + z_2)|$.

Sol. As given, let

$$\frac{2z_1}{3z_2} = iy \quad \text{or} \quad \frac{z_1}{z_2} = \frac{3}{2}iy$$

$$\text{so that } \left| \frac{z_1 - z_2}{z_1 + z_2} \right| = \left| \frac{\frac{z_1}{z_2} - 1}{\frac{z_1}{z_2} + 1} \right| = \left| \frac{\frac{3}{2}iy - 1}{\frac{3}{2}iy + 1} \right| = \left| \frac{1 - \frac{3}{2}iy}{1 + \frac{3}{2}iy} \right| = 1$$

$$[\because |z| = |\bar{z}|]$$

ILLUSTRATION 3.57

If complex numbers satisfy the system of equations $z^3 + \bar{\omega}^7 = 0$ and $z^5 \omega^{11} = 1$, then find z .

$$\text{Sol. } z^3 + \bar{\omega}^7 = 0$$

$$\Rightarrow z^3 = -\bar{\omega}^7$$

$$\Rightarrow |z|^3 = |-\bar{\omega}^7| = |\omega|^7$$

$$\Rightarrow |z|^{15} = |\omega|^{35} \quad (1)$$

$$\text{Also, } z^5 \omega^{11} = 1$$

$$\Rightarrow |z|^5 |\omega|^{11} = 1$$

$$\Rightarrow |z|^{15} |\omega|^{33} = 1 \quad (2)$$

From (1) and (2), we have

$$|z| = |\omega| = 1$$

$$\text{Again, } \bar{\omega}^7 = -z^3 \quad \text{and} \quad \omega^{11} = z^{-5}$$

$$\Rightarrow \bar{\omega}^{77} \cdot \omega^{77} = -z^{33} \cdot z^{-35}$$

$$\Rightarrow z^2 = -1 = i^2 \Rightarrow z = \pm i$$

CONCEPT APPLICATION EXERCISE 3.6

- For $z_1 = \sqrt[6]{(1-i)/(1+i\sqrt{3})}$, $z_2 = \sqrt[6]{(1-i)/(\sqrt{3}+i)}$, $z_3 = \sqrt[6]{(1+i)/(\sqrt{3}-i)}$, prove that $|z_1| = |z_2| = |z_3|$.
- If $\sqrt{3} + i = (a + ib)(c + id)$, then find the value of $\tan^{-1}(b/a) + \tan^{-1}(d/c)$.
- If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find the value of $\arg(z_1/z_4) + \arg(z_2/z_3)$.
- Find the modulus, argument, and the principal argument of the complex numbers $\frac{i-1}{i\left(1-\cos\frac{2\pi}{5}\right) + \sin\frac{2\pi}{5}}$.
- If $(1+i)(1+2i)(1+3i) \cdots (1+ni) = (x+iy)$, then show that $2 \times 5 \times 10 \times \cdots \times (1+n^2) = x^2 + y^2$.
- If $a + ib = \frac{(x+i)^2}{2x+1}$, prove that $a^2 + b^2 = \frac{(x+1)^2}{(2x+1)^2}$.
- Let z be a complex number satisfying the equation $(z^3 + 3)^2 = -16$, then find the value of $|z|$.
- If θ is real and z_1, z_2 are connected by $z_1^2 + z_2^2 + 2z_1z_2 \cos \theta = 0$, then prove that the triangle formed by vertices O, z_1 , and z_2 is isosceles.
- If $|z_1 - z_0| = |z_2 - z_0| = a$ and $\arg((z_2 - z_0)/(z_0 - z_1)) = \pi/2$, then find z_0 .
- If $\left| \frac{z-2}{z-3} \right| = 2$ represents a circle, then find its centre and radius.

ANSWERS

$$2. \quad n\pi + \frac{\pi}{6}, n \in \mathbb{Z}$$

$$3. \quad 0$$

$$4. \quad \text{Modulus} = \frac{1}{\sqrt{2}} \operatorname{cosec} \frac{\pi}{5}, \text{ argument} = \frac{11\pi}{20}$$

$$7. \quad 5^{1/3}$$

$$9. \quad \frac{1}{2} \{(i+1)z_1 + (1-i)z_2\}$$

$$10. \quad \text{Centre} \equiv (10/3, 0); \text{ Radius} = 2/3$$

DE MOIVRE'S THEOREM

Statement:

- (i) If $n \in \mathbb{Z}$ (the set of integers), then
 $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$
- (ii) If $n \in \mathbb{Q}$ (the set of rational numbers), then $\cos n\theta + i \sin n\theta$ is one of the values of $(\cos \theta + i \sin \theta)^n$.

Proof:

- (i) When $n \in \mathbb{Z}$, we know that
 $e^{i\theta} = \cos \theta + i \sin \theta$
 or $(e^{i\theta})^n = (\cos \theta + i \sin \theta)^n$
 or $e^{i(n\theta)} = (\cos \theta + i \sin \theta)^n$
 or $\cos(n\theta) + i \sin(n\theta) = (\cos \theta + i \sin \theta)^n$
- (ii) Let n be a rational number. Let $n = p/q$, where p, q are integers and $q \neq 0$. From part (i), we have
- $$\left(\cos \frac{p\theta}{q} + i \sin \frac{p\theta}{q} \right)^q = \cos \left(\left(\frac{p\theta}{q} \right) q \right) + i \sin \left(\left(\frac{p\theta}{q} \right) q \right)$$
- $$= \cos p\theta + i \sin p\theta$$
- $$\Rightarrow \cos \frac{p\theta}{q} + i \sin \frac{p\theta}{q} \text{ is one of the values of } (\cos p\theta + i \sin p\theta)^{1/q}$$
- $$\Rightarrow \cos \frac{p\theta}{q} + i \sin \frac{p\theta}{q} \text{ is one of the values of } [(\cos \theta + i \sin \theta)^p]^{1/q}$$
- $$\Rightarrow \cos \frac{p\theta}{q} + i \sin \frac{p\theta}{q} \text{ is one of the values of } (\cos \theta + i \sin \theta)^{p/q}$$

Note:

- De Moivre's theorem is also true for $(\cos \theta - i \sin \theta)$, i.e.,
 $(\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta$, because
 $(\cos \theta - i \sin \theta)^n = [\cos(-\theta) + i \sin(-\theta)]^n$
 $= \cos(-n\theta) + i \sin(-n\theta)$
 $= \cos n\theta - i \sin n\theta$.
- $\frac{1}{\cos \theta + i \sin \theta} = (\cos \theta + i \sin \theta)^{-1} = \cos \theta - i \sin \theta$
- $(\sin \theta \pm i \cos \theta)^n \neq \sin n\theta \pm i \cos n\theta$.
- $(\sin \theta + i \cos \theta)^n = [\cos(\pi/2 - \theta) + i \sin(\pi/2 - \theta)]^n$
 $= \cos(n\pi/2 - n\theta) + i \sin(n\pi/2 - n\theta)$
- $(\cos \theta + i \sin \phi)^n \neq \cos n\theta + i \sin n\phi$

ILLUSTRATION 3.58

Express the following in $a + ib$ form:

- (i) $\left(\frac{\cos \theta + i \sin \theta}{\sin \theta + i \cos \theta} \right)^4$
- (ii) $\frac{(\cos 2\theta - i \sin 2\theta)^4 (\cos 4\theta + i \sin 4\theta)^{-5}}{(\cos 3\theta + i \sin 3\theta)^{-2} (\cos 3\theta - i \sin 3\theta)^{-9}}$
- (iii) $\frac{(\sin \pi/8 + i \cos \pi/8)^8}{(\sin \pi/8 - i \cos \pi/8)^8}$

Sol.

(i) $\left(\frac{\cos \theta + i \sin \theta}{\sin \theta + i \cos \theta} \right)^4 = \frac{(\cos \theta + i \sin \theta)^4}{i^4 (\cos \theta - i \sin \theta)^4}$

$$= \frac{(\cos \theta + i \sin \theta)^4}{(\cos \theta + i \sin \theta)^{-4}}$$

$$= (\cos 8\theta + i \sin 8\theta)^8$$

$$= \cos 8\theta + i \sin 8\theta$$

(ii) $\frac{(\cos 2\theta - i \sin 2\theta)^4 (\cos 4\theta + i \sin 4\theta)^{-5}}{(\cos 3\theta + i \sin 3\theta)^{-2} (\cos 3\theta - i \sin 3\theta)^{-9}}$

$$= \frac{[(\cos \theta + i \sin \theta)^{-2}]^4 [(\cos \theta + i \sin \theta)^4]^{-5}}{[(\cos \theta + i \sin \theta)^3]^{-2} [(\cos \theta + i \sin \theta)^{-3}]^{-9}}$$

$$= \frac{(\cos \theta + i \sin \theta)^{-8} (\cos \theta + i \sin \theta)^{-20}}{(\cos \theta + i \sin \theta)^{-6} (\cos \theta + i \sin \theta)^{27}}$$

$$= (\cos \theta + i \sin \theta)^{-8-20+6-27}$$

$$= (\cos \theta + i \sin \theta)^{-49}$$

$$= \cos 49\theta - i \sin 49\theta$$

(iii) $\frac{(\sin \pi/8 + i \cos \pi/8)^8}{(\sin \pi/8 - i \cos \pi/8)^8}$

$$\frac{i^8 (\cos \pi/8 - i \sin \pi/8)^8}{(-i)^8 (\cos \pi/8 + i \sin \pi/8)^8} = \frac{\cos \pi - i \sin \pi}{\cos \pi + i \sin \pi} = 1$$

ILLUSTRATION 3.59

If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2} \right)^5$, then prove that $\text{Im}(z) = 0$.

Sol. Given that

$$z = \left(\frac{\sqrt{3}}{2} + i \frac{1}{2} \right)^5 + \left(\frac{\sqrt{3}}{2} - i \frac{1}{2} \right)^5$$

$$= \left[\cos \left(\frac{\pi}{6} \right) + i \sin \left(\frac{\pi}{6} \right) \right]^5 + \left[\cos \left(\frac{\pi}{6} \right) - i \sin \left(\frac{\pi}{6} \right) \right]^5$$

$$= \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} + \cos \frac{5\pi}{6} - i \sin \frac{5\pi}{6}$$

Hence, $\text{Im}(z) = 0$.

ILLUSTRATION 3.60

Prove that the roots of the equation $x^4 - 2x^2 + 4 = 0$ forms a rectangle.

Sol. $x^4 - 2x^2 + 4 = 0$

$$\therefore x^2 = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{3}i = 2 \frac{1 \pm \sqrt{3}i}{2}$$

$$= 2 \left(\cos \frac{\pi}{3} \pm i \sin \frac{\pi}{3} \right)$$

$$\therefore x = \pm \sqrt{2} \left(\cos \frac{\pi}{3} \pm i \sin \frac{\pi}{3} \right)^{1/2}$$

$$= \pm\sqrt{2}\left(\cos\frac{\pi}{6} \pm i\sin\frac{\pi}{6}\right)$$

$$\therefore x = \sqrt{2}e^{i\pi/6}, -\sqrt{2}e^{i\pi/6}, \sqrt{2}e^{-i\pi/6}, -\sqrt{2}e^{-i\pi/6}$$

Clearly these points form rectangle inscribed in circle of radius $\sqrt{2}$.

ILLUSTRATION 3.61

If $z + 1/z = 2 \cos \theta$, prove that $|(z^{2n} - 1) / (z^{2n} + 1)| = |\tan n\theta|$

Sol. $z + \frac{1}{z} = 2 \cos \theta$

or $z^2 - 2 \cos \theta z + 1 = 0$

or $z = \frac{2 \cos \theta \pm \sqrt{4 \cos^2 \theta - 4}}{2}$

$$= \cos \theta \pm i \sin \theta$$

Taking positive sign, we get

$$z = \cos \theta + i \sin \theta$$

$$\therefore \frac{1}{z} = (\cos \theta - i \sin \theta)$$

$$\therefore \frac{z^{2n} - 1}{z^{2n} + 1} = \frac{z^n - \frac{1}{z^n}}{z^n + \frac{1}{z^n}}$$

$$\begin{aligned} &= \frac{(\cos \theta + i \sin \theta)^n - (\cos \theta - i \sin \theta)^n}{(\cos \theta + i \sin \theta)^n + (\cos \theta - i \sin \theta)^n} \\ &= \frac{2i \sin n\theta}{2 \cos n\theta} \\ &= i \tan n\theta \end{aligned}$$

Taking negative sign, we get

$$\frac{z^{2n} - 1}{z^{2n} + 1} = \frac{-2i \sin n\theta}{2 \cos n\theta} = -i \tan n\theta$$

$$\Rightarrow \left| \frac{z^{2n} - 1}{z^{2n} + 1} \right| = |\pm i \tan n\theta| = \tan n\theta$$

ILLUSTRATION 3.62

If $z = x + iy$ is a complex number with $x, y \in \mathbb{Q}$ and $|z| = 1$, then show that $|z^{2n} - 1|$ is a rational number for every $n \in \mathbb{N}$.

Sol. $|z| = 1$

$$\Rightarrow z = e^{i\theta} = x + iy$$

$$\Rightarrow x = \cos \theta, y = \sin \theta$$

Now $\cos \theta$ and $\sin \theta \in \mathbb{Q}$. Also,

$$\begin{aligned} |z^{2n} - 1|^2 &= (z^{2n} - 1)(\bar{z}^{2n} - 1) \\ &= (z\bar{z})^{2n} - z^{2n} - \bar{z}^{2n} + 1 \\ &= 2 - (z^{2n} + \bar{z}^{2n}) \\ &= 2 - 2 \cos 2n\theta = 4 \sin^2 n\theta \end{aligned}$$

$$\Rightarrow |z^{2n} - 1| = 2 |\sin n\theta|$$

Now, $\sin n\theta = {}^nC_1 \cos^{n-1} \theta \sin \theta - {}^nC_3 \cos^{n-3} \theta \sin^3 \theta + \dots$
 $= \text{Rational number} \quad (\because \sin \theta, \cos \theta \text{ are rationals})$

$$\Rightarrow |z^{2n} - 1| = \text{Rational number}$$

ILLUSTRATION 3.63

If $z = \cos \theta + i \sin \theta$ is a root of the equation $a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_{n-1} z + a_n = 0$, then prove that

(i) $a_0 + a_1 \cos \theta + a_2 \cos 2\theta + \dots + a_n \cos n\theta = 0$

(ii) $a_1 \sin \theta + a_2 \sin 2\theta + \dots + a_n \sin n\theta = 0$

Sol. Dividing the given equation by z^n , we get

$$a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_{n-1} z^{1-n} + a_n z^{-n} = 0$$

Now, $z = \cos \theta + i \sin \theta = e^{i\theta}$ satisfies the above equation. Hence,

$$a_0 + a_1 e^{-i\theta} + a_2 e^{-2i\theta} + \dots + a_{n-1} e^{-i(n-1)\theta} + a_n e^{-in\theta} = 0$$

$$\Rightarrow (a_0 + a_1 \cos \theta + a_2 \cos 2\theta + \dots + a_n \cos n\theta)$$

$$- i(a_1 \sin \theta + a_2 \sin 2\theta + \dots + a_n \sin n\theta) = 0$$

$$\Rightarrow a_0 + a_1 \cos \theta + a_2 \cos 2\theta + \dots + a_n \cos n\theta = 0$$

and $a_1 \sin \theta + a_2 \sin 2\theta + \dots + a_n \sin n\theta = 0$

CONCEPT APPLICATION EXERCISE 3.7

1. Express the following in $a + ib$ form:

(a) $\frac{(\cos \alpha + i \sin \alpha)^4}{(\sin \beta + i \cos \beta)^5}$ (b) $\left(\frac{1 + \cos \phi + i \sin \phi}{1 + \cos \phi - i \sin \phi} \right)^n$

(c) $\frac{(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta)}{(\cos \gamma + i \sin \gamma)(\cos \delta + i \sin \delta)}$

2. Find the value of the following expression:

$$\left[\frac{1 - \cos \frac{\pi}{10} + i \sin \frac{\pi}{10}}{1 - \cos \frac{\pi}{10} - i \sin \frac{\pi}{10}} \right]^{10}$$

3. If $iz^4 + 1 = 0$, then prove that z can take the value $\cos \pi/8 + i \sin \pi/8$.

4. If n is a positive integer, then prove that

$$(1 + i)^n + (1 - i)^n = (\sqrt{2})^{n+2} \cos \left(\frac{n\pi}{4} \right)$$

5. If $z = (a + ib)^5 + (b + ia)^5$, then prove that $\operatorname{Re}(z) = \operatorname{Im}(z)$, where $a, b \in \mathbb{R}$.

6. If $\cos \alpha + \cos \beta + \cos \gamma = 0$ and also $\sin \alpha + \sin \beta + \sin \gamma = 0$, then prove that

(a) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

(b) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin (\alpha + \beta + \gamma)$

(c) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos (\alpha + \beta + \gamma)$

ANSWERS

1. (a) $\sin (4\alpha + 5\beta) - i \cos (4\alpha + 5\beta)$

(b) $\cos n\phi + i \sin n\phi$

(c) $\cos(\alpha + \beta - \gamma - \delta) + i \sin(\alpha + \beta - \gamma - \delta)$

2. -1

IMPORTANT PROPERTIES OF MODULUS

There are some important properties of modulus which can be helpful to solve many problems involving complex numbers. Let us study these properties.

Property 1:

$$|z|^2 = z \bar{z}$$

Clearly, for $z = x + iy$,

$$z\bar{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2 = |z|^2$$

ILLUSTRATION 3.64

If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$, and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, then find the value of $|z_1 + z_2 + z_3|$.

Sol. $|z_1| = 1 \Rightarrow z_1\bar{z}_1 = 1$, $|z_2| = 2 \Rightarrow z_2\bar{z}_2 = 4$, $|z_3| = 3 \Rightarrow z_3\bar{z}_3 = 9$

Also, $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$

or $|z_1z_2z_3\bar{z}_3 + z_1z_2z_3\bar{z}_2 + z_1\bar{z}_1z_2z_3| = 12$

or $|z_1z_2z_3| |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

or $|z_1| |z_2| |z_3| |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

or $6 |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

or $|\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 2$

or $|z_1 + z_2 + z_3| = 2$

ILLUSTRATION 3.65

If α and β are different complex numbers with $|\beta| = 1$, then find the value of $|(\beta - \alpha) / (1 - \bar{\alpha}\beta)|$.

Sol. Given, $|\beta| = 1$

$$\Rightarrow \beta\bar{\beta} = 1 \text{ or } \beta = \frac{1}{\bar{\beta}} \quad (1)$$

$$\begin{aligned} \text{Now, } \left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| &= \left| \frac{\frac{1}{\bar{\beta}} - \alpha}{1 - \bar{\alpha}\beta} \right| = \left| \frac{1}{\bar{\beta}} \right| \left| \frac{1 - \alpha\bar{\beta}}{1 - \bar{\alpha}\beta} \right| = \frac{1}{|\beta|} \left| \frac{1 - \alpha\bar{\beta}}{1 - \bar{\alpha}\beta} \right| \\ &= \left| \frac{1 - \bar{\alpha}\beta}{1 - \bar{\alpha}\beta} \right| = 1 \end{aligned}$$

Property 2:

$$(i) |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2)$$

$$(ii) |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1\bar{z}_2)$$

Proof:

$$\begin{aligned} (i) |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \\ &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_1\bar{z}_2 \\ &= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2) \quad (\because \bar{z}_1z_2 = \overline{z_1\bar{z}_2}) \end{aligned}$$

Also, if $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$, then

$$\begin{aligned} |z_1 + z_2|^2 &= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(|z_1||z_2|e^{i(\theta_1 - \theta_2)}) \\ &= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2) \\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \end{aligned}$$

Thus, $|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$

Or $|z_1 + z_2| \leq |z_1| + |z_2|$

Replacing z_1 by $-z_2$, we get

$$|z_1 - z_2| \leq |z_1| + |z_2|$$

Similarly, we can prove the result (ii) and hence,

$$|z_1 \pm z_2| \geq ||z_1| - |z_2||.$$

ILLUSTRATION 3.66

Prove that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$, if z_1/z_2 is purely imaginary.

Sol. Given

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$$

$$\text{or } |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1z_2 = |z_1|^2 + |z_2|^2$$

$$\text{or } z_1\bar{z}_2 + \bar{z}_1z_2 = 0$$

$$\text{or } \frac{z_1}{z_2} + \frac{\bar{z}_1}{\bar{z}_2} = 0$$

$$\text{or } \frac{z_1}{z_2} + \left(\frac{z_1}{z_2} \right) = 0$$

Hence, $\frac{z_1}{z_2}$ is purely imaginary

ILLUSTRATION 3.67

Let $|(\bar{z}_1 - 2\bar{z}_2) / (2 - z_1\bar{z}_2)| = 1$ and $|z_2| \neq 1$, where z_1 and z_2 are complex numbers. Show that $|z_1| = 2$.

$$\text{Sol. } \left| \frac{\bar{z}_1 - 2\bar{z}_2}{2 - z_1\bar{z}_2} \right| = 1$$

$$\text{or } |\bar{z}_1 - 2\bar{z}_2|^2 = |2 - z_1\bar{z}_2|^2$$

$$\text{or } (\bar{z}_1 - 2\bar{z}_2)(\overline{\bar{z}_1 - 2\bar{z}_2}) = (2 - z_1\bar{z}_2)(\overline{2 - z_1\bar{z}_2})$$

$$\text{or } (\bar{z}_1 - 2\bar{z}_2)(z_1 - 2z_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1z_2)$$

$$\text{or } z_1\bar{z}_1 - 2\bar{z}_1z_2 - 2z_1\bar{z}_2 + 4z_2\bar{z}_2 = 4 - 2\bar{z}_1z_2 - 2z_1\bar{z}_2 + z_1\bar{z}_1z_2\bar{z}_2$$

$$\text{or } |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2|z_2|^2$$

$$\text{or } |z_1|^2 - |z_1|^2|z_2|^2 + 4|z_2|^2 - 4 = 0$$

$$\text{or } |z_1|^2(1 - |z_2|^2) + 4(|z_2|^2 - 1) = 0$$

$$\text{or } (|z_2|^2 - 1)(|z_1|^2 - 4) = 0$$

$$\text{or } |z_1| = 2 \quad (\text{as } |z_2| \neq 1)$$

ILLUSTRATION 3.68

If z_1 and z_2 are two complex numbers and $c > 0$, then prove that $|z_1 + z_2|^2 \leq (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$.

Sol. We have to prove

$$|z_1 + z_2|^2 \leq (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$$

$$\text{or } |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1z_2 \leq (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$$

$$\text{or } z_1\bar{z}_2 + \bar{z}_1z_2 \leq c|z_1|^2 + c^{-1}|z_2|^2$$

$$\text{or } c|z_1|^2 + \frac{1}{c}|z_2|^2 - z_1\bar{z}_2 - \bar{z}_1z_2 \geq 0 \quad [\text{using } \operatorname{Re}(z_1\bar{z}_2) \leq |z_1\bar{z}_2|]$$

$$\text{or } \left| \sqrt{c}z_1 - \frac{1}{\sqrt{c}}z_2 \right|^2 \geq 0$$

which is always true.

Property 3:

$|z_1 - z_2|$ = distance between two complex numbers $A(z_1)$ and $B(z_2)$ on the Argand plane.

Proof:

$$\begin{aligned} |z_1 - z_2| &= |(x_1 + iy_1) - (x_2 + iy_2)| \\ &= |(x_1 - x_2) + i(y_1 - y_2)| \\ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \text{Distance between points } A(x_1, y_1) \text{ and } B(x_2, y_2) \end{aligned}$$

Also, $|z_1 + z_2| = |z_1 - (-z_2)|$ = distance between complex numbers z_1 and $-z_2$.

ILLUSTRATION 3.69

If z_1, z_2, z_3, z_4 are the affixes of four points in the Argand plane, z is the affix of a point such that $|z - z_1| = |z - z_2| = |z - z_3| = |z - z_4|$, then prove that z_1, z_2, z_3, z_4 are concyclic.

Sol. We have,

$$|z - z_1| = |z - z_2| = |z - z_3| = |z - z_4|$$

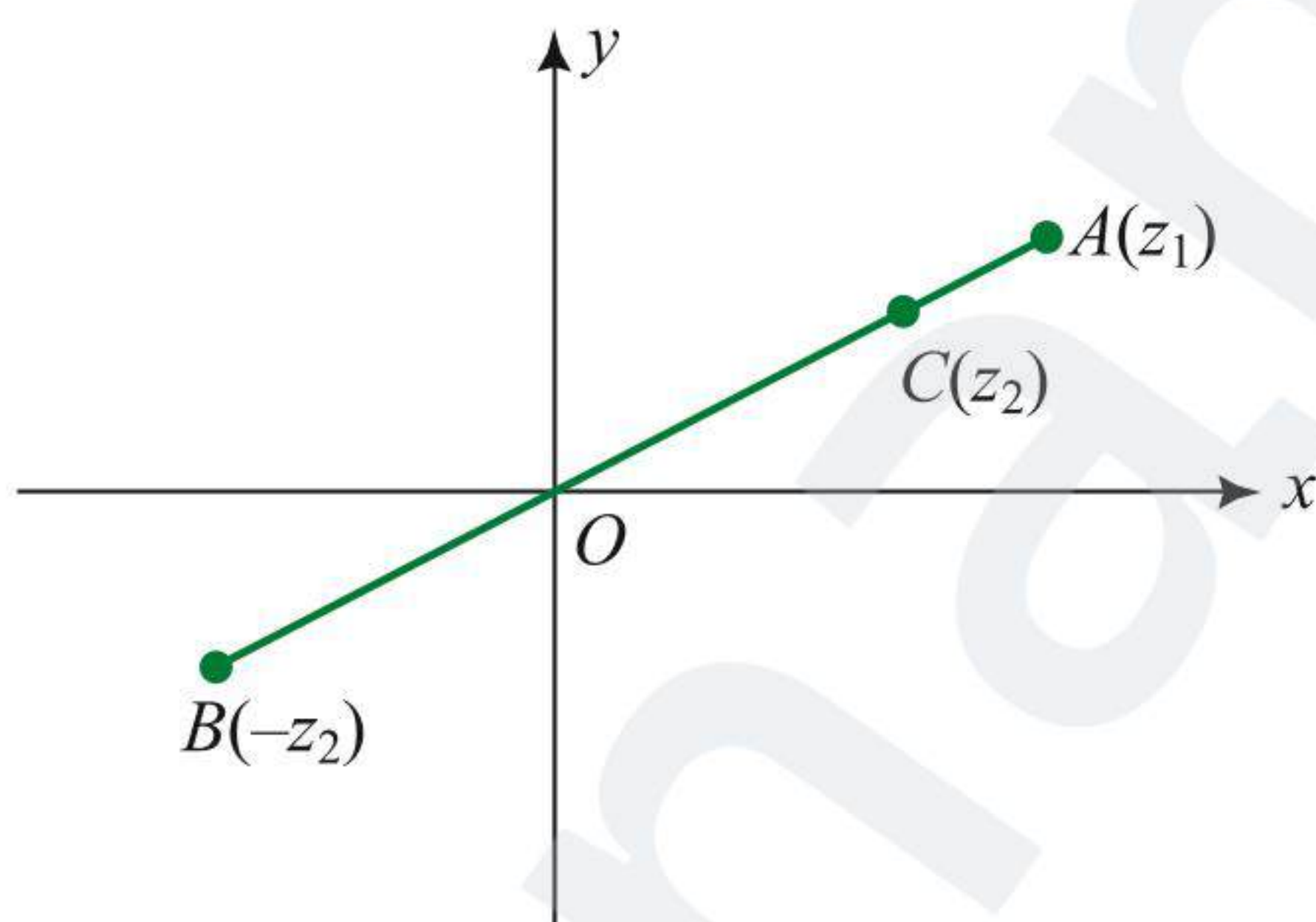
Therefore, the point having affix z is equidistant from the four points having affixes z_1, z_2, z_3, z_4 . Thus, z is the affix of either the center of a circle or the point of intersection of diagonals of a rectangle. Therefore, z_1, z_2, z_3, z_4 are concyclic.

ILLUSTRATION 3.70

- (i) If $|z_1 + z_2| = |z_1| + |z_2|$, then prove that $\arg(z_1) = \arg(z_2)$
(ii) If $|z_1 - z_2| = |z_1| + |z_2|$, then prove that $\arg(z_1) - \arg(z_2) = \pi$

Sol.

- (i) $|z_1 + z_2| = |z_1| + |z_2|$
 $\Rightarrow |z_1 + (-z_2)| = |z_1| + |-z_2|$
 $\Rightarrow AB = AO + OB$ as shown in the figure

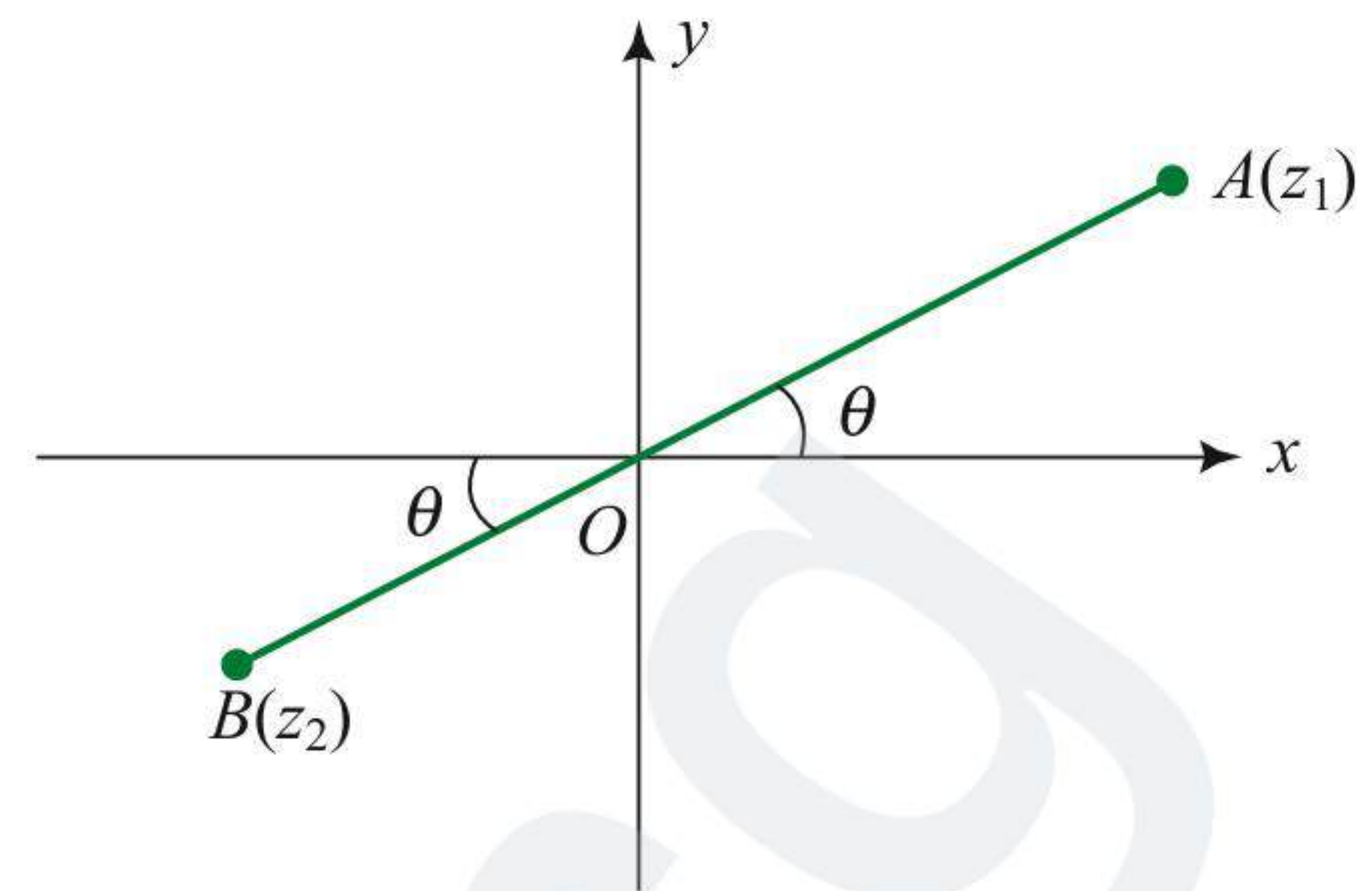


- $\Rightarrow$ Points $A(z_1), O(0), B(-z_2)$ are collinear
 $\Rightarrow A(z_1), O(0)$ and $C(z_2)$ are collinear as shown in figure
 $\Rightarrow \arg(z_1) = \arg(z_2)$

Alternate method:

$$\begin{aligned} |z_1 + z_2| &= |z_1| + |z_2| \\ \text{or } |z_1 + z_2|^2 &= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\ \text{or } |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2) &= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\ \text{or } 2\operatorname{Re}(z_1\bar{z}_2) &= 2|z_1||z_2| \\ \Rightarrow \cos(\theta_1 - \theta_2) &= 1 \\ \Rightarrow \theta_1 - \theta_2 &= 0 \\ \Rightarrow \arg(z_1) &= \arg(z_2) \end{aligned}$$

- (ii) $|z_1 - z_2| = |z_1| + |z_2|$
 $\Rightarrow AB = AO + OB$ as shown in the figure.



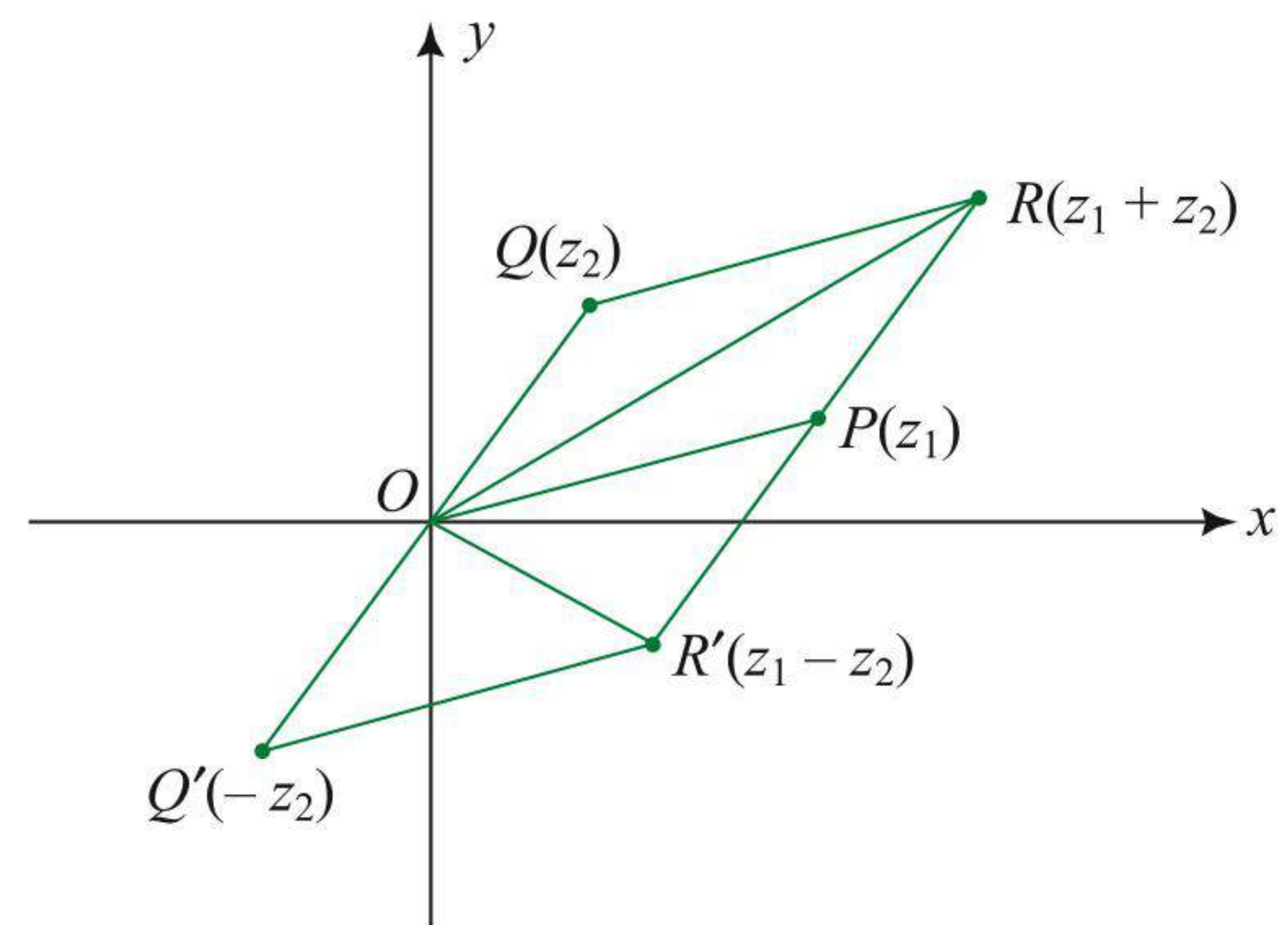
- $\Rightarrow$ Points $z_1, O(\text{origin}), z_2$ are collinear as shown in figure
If $\arg(z_1) = \theta$, then $\arg(z_2) = \theta - \pi$
 $\Rightarrow \arg(z_1) - \arg(z_2) = \pi$

Alternate method:

$$\begin{aligned} |z_1 - z_2| &= |z_1| + |z_2| \\ \text{or } |z_1 - z_2|^2 &= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\ \text{or } |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1\bar{z}_2) &= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\ \text{or } -2\operatorname{Re}(z_1\bar{z}_2) &= 2|z_1||z_2| \\ \Rightarrow \cos(\theta_1 - \theta_2) &= -1 \\ \Rightarrow \theta_1 - \theta_2 &= \pi \end{aligned}$$

Property 4:

Geometrical representation of addition and subtraction of complex numbers (Triangular Inequalities):



Let z_1 and z_2 be two complex numbers represented by points P and Q , respectively.

Now $OP = |z_1|$ and $OQ = |z_2|$

Let R represents the complex number $z_1 + z_2$.

Now PR = Distance between complex numbers $(z_1 + z_2)$ and z_1
 $= |(z_1 + z_2) - z_1|$
 $= |z_2| = OQ$

Similarly, $QR = |(z_1 + z_2) - z_2| = |z_1| = OP$

Hence, points O, P, R, Q completes the parallelogram.

Also in triangle OPR , we have $OP + PR > OR$

$$\Rightarrow |z_1| + |z_2| > |z_1 + z_2|$$

In above result replacing z_2 by $-z_2$ we get

$$|z_1| + |-z_2| > |z_1 - z_2|$$

$$\text{or } |z_1| + |z_2| > |z_1 - z_2|$$

Also when z_1, O, z_2 are collinear, we have

$$|z_1 + z_2| = |z_1| + |z_2|$$

or $|z_1 - z_2| = |z_1| + |z_2|$

Thus, we have $|z_1 \pm z_2| \leq |z_1| + |z_2|$

In the figure, $OPR'Q'$ is a parallelogram. Thus,

$$OP = Q'R' = |z_1| \text{ and } OQ = OQ' = PR' = |z_2|$$

Also in triangle OPR' , we have $|OP - PR'| < OR'$

$$\Rightarrow ||z_1| - |z_2|| < |z_1 - z_2|$$

In above result, replacing z_2 by $-z_2$, we get

$$||z_1| - |-z_2|| < |z_1 + z_2|$$

or $||z_1| - |z_2|| < |z_1 + z_2|$

Also when z_1, O, z_2 are collinear, we have

$$|z_1 + z_2| = |z_1| - |z_2|$$

or $|z_1 - z_2| = |z_1| - |z_2|$

Thus, we have $|z_1 \pm z_2| \geq ||z_1| - |z_2||$

Note:

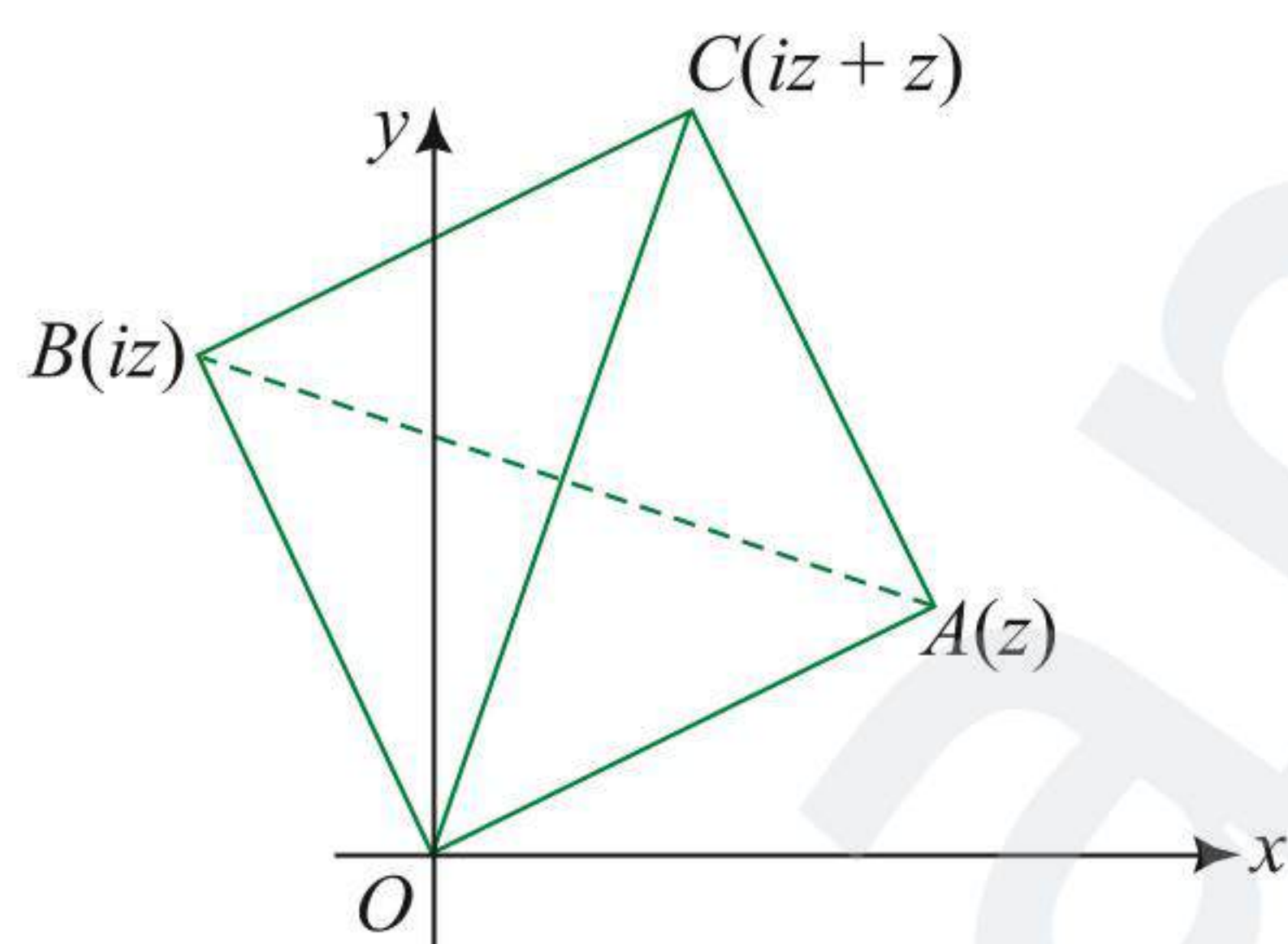
$$|z_1 + z_2 + z_3| \leq |z_1 + z_2| + |z_3| \leq |z_1| + |z_2| + |z_3|$$

$$\text{In general, } |z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|$$

ILLUSTRATION 3.71

Show that the area of the triangle on the Argand diagram formed by the complex numbers z, iz , and $z + iz$ is $(1/2)|z|^2$.

Sol. Let the vertices of the triangle be $A(z), B(iz), C(z + iz)$. We know that iz is obtained by rotating OA through an angle 90° in anticlockwise direction. Also, point $z + iz$ can be obtained by completing the parallelogram two of whose adjacent sides are OA and OB . From Argand diagram, it is clear that



$$\text{Area of } \triangle ABC = \text{Area of } \triangle OAB$$

$$= \frac{1}{2} \times OA \times OB \quad [\because \text{it is right angled at point } O]$$

$$= \frac{1}{2} |z| \times |iz|$$

$$= \frac{1}{2} |z|^2$$

ILLUSTRATION 3.72

Find the minimum value of $|z - 1|$ if $||z - 3| - |z + 1|| = 2$.

Sol. $2 = ||z - 3| - |z + 1|| \leq |(z - 3) + (z + 1)|$

$$\Rightarrow |2z - 2| \geq 2$$

$$\Rightarrow |z - 1| \geq 1$$

Thus, minimum value of $|z - 1|$ is 1.

ILLUSTRATION 3.73

Find the greatest and the least value of $|z_1 + z_2|$ if $z_1 = 24 + 7i$ and $|z_2| = 6$.

Sol. $|z_1 + z_2| \leq |z_1| + |z_2|$
 $= |24 + 7i| + 6$
 $= 25 + 6 = 31$

Also, $|z_1 + z_2| = |z_1 - (-z_2)| \geq ||z_1| - |z_2||$

$$\Rightarrow |z_1 + z_2| \geq |25 - 6| = 19$$

Hence, the least value of $|z_1 + z_2|$ is 19 and the greatest value is 25.

ILLUSTRATION 3.74

If z is a complex number, then find the minimum value of $|z| + |z - 1| + |2z - 3|$.

Sol. $E = |z| + |z - 1| + |2z - 3|$
 $= |z| + |z - 1| + |3 - 2z| \geq |z + z - 1 + 3 - 2z| = 2$

$$\therefore |z| + |z - 1| + |2z - 3| \geq 2$$

Therefore, minimum value of E is 2.

ILLUSTRATION 3.75

If $|z_1 - 1| \leq 1, |z_2 - 2| \leq 2, |z_3 - 3| \leq 3$, then find the greatest value of $|z_1 + z_2 + z_3|$.

Sol. $|z_1 + z_2 + z_3| = |(z_1 - 1) + (z_2 - 2) + (z_3 - 3) + 6|$
 $\leq |z_1 - 1| + |z_2 - 2| + |z_3 - 3| + 6$
 $\leq 1 + 2 + 3 + 6 = 12$

Hence, the greatest value is 12.

ILLUSTRATION 3.76

Prove the following inequalities:

(i) $\left| \frac{z}{|z|} - 1 \right| \leq |\arg z|$ (ii) $|z - 1| \leq |z| |\arg z| + ||z| - 1|$

Sol.

(i) Let $z = |z|(\cos \theta + i \sin \theta)$

$$\Rightarrow \left| \frac{z}{|z|} - 1 \right| = |\cos \theta + i \sin \theta - 1|$$

$$= \sqrt{(\cos \theta - 1)^2 + \sin^2 \theta}$$

$$= \sqrt{2 - 2 \cos \theta}$$

$$= \sqrt{4 \sin^2 \frac{\theta}{2}}$$

$$= 2 \left| \sin \frac{\theta}{2} \right|$$

$$\leq 2 \left| \frac{\theta}{2} \right| \quad (\because |\sin \theta| \leq |\theta|)$$

$$= |\theta| = |\arg z| \quad (1)$$

$$\begin{aligned}
 \text{(ii)} \quad |z - 1| &= |z - |z| + |z| - 1| \\
 &\leq |z - |z|| + ||z| - 1| \\
 &= |z| \left| \frac{z}{|z|} - 1 \right| + ||z| - 1| \\
 &\leq |z| |\arg z| + ||z| - 1| \quad [\text{From (1)}]
 \end{aligned}$$

CONCEPT APPLICATION EXERCISE 3.8

1. a, b, c are three complex numbers on the unit circle $|z| = 1$, such that $abc = a + b + c$. Then find the value of $|ab + bc + ca|$.
2. Let z be not a real number such that $(1 + z + z^2) / (1 - z + z^2) \in \mathbb{R}$, then prove that $|z| = 1$.
3. If z_1, z_2, z_3 are distinct nonzero complex numbers and $a, b, c \in \mathbb{R}^+$ such that $\frac{a}{|z_1 - z_2|} = \frac{b}{|z_2 - z_3|} = \frac{c}{|z_3 - z_1|}$, then find the value of $\frac{a^2}{z_1 - z_2} + \frac{b^2}{z_2 - z_3} + \frac{c^2}{z_3 - z_1}$.
4. If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$, then prove that $|(1 - z_1 \bar{z}_2) / (z_1 - z_2)| < 1$.
5. (a) If $|z_1 - z_2| = |z_1| - |z_2|$, then prove that $\arg(z_1) = \arg(z_2)$
(b) If $|z_1 + z_2| = |z_1| - |z_2|$, then prove that $\arg(z_1) - \arg(z_2) = \pi$
6. For any complex number z , find the minimum value of $|z| + |z - 2i|$.
7. If z is any complex number such that $|z + 4| \leq 3$, then find the greatest value of $|z + 1|$.
8. $Z \in \mathbb{C}$ satisfies the condition $|Z| \geq 3$. Then find the least value of $\left| Z + \frac{1}{Z} \right|$.
9. If a, b, c are nonzero complex numbers of equal moduli and satisfy $az^2 + bz + c = 0$, then prove that $(\sqrt{5} - 1) / 2 \leq |z| \leq (\sqrt{5} + 1) / 2$.
10. If $|z| \leq 4$, then find the maximum value of $|iz + 3 - 4i|$.
11. Let $z_1, z_2, z_3, \dots, z_n$ be the complex numbers such that $|z_1| = |z_2| = \dots = |z_n| = 1$.
If $z = \left(\sum_{k=1}^n z_k \right) \left(\sum_{k=1}^n \frac{1}{z_k} \right)$ then prove that
(a) z is a real number (b) $0 < z \leq n^2$

ANSWERS

1. 1 3. 0 6. 2 7. 6 8. 8/3
10. 9

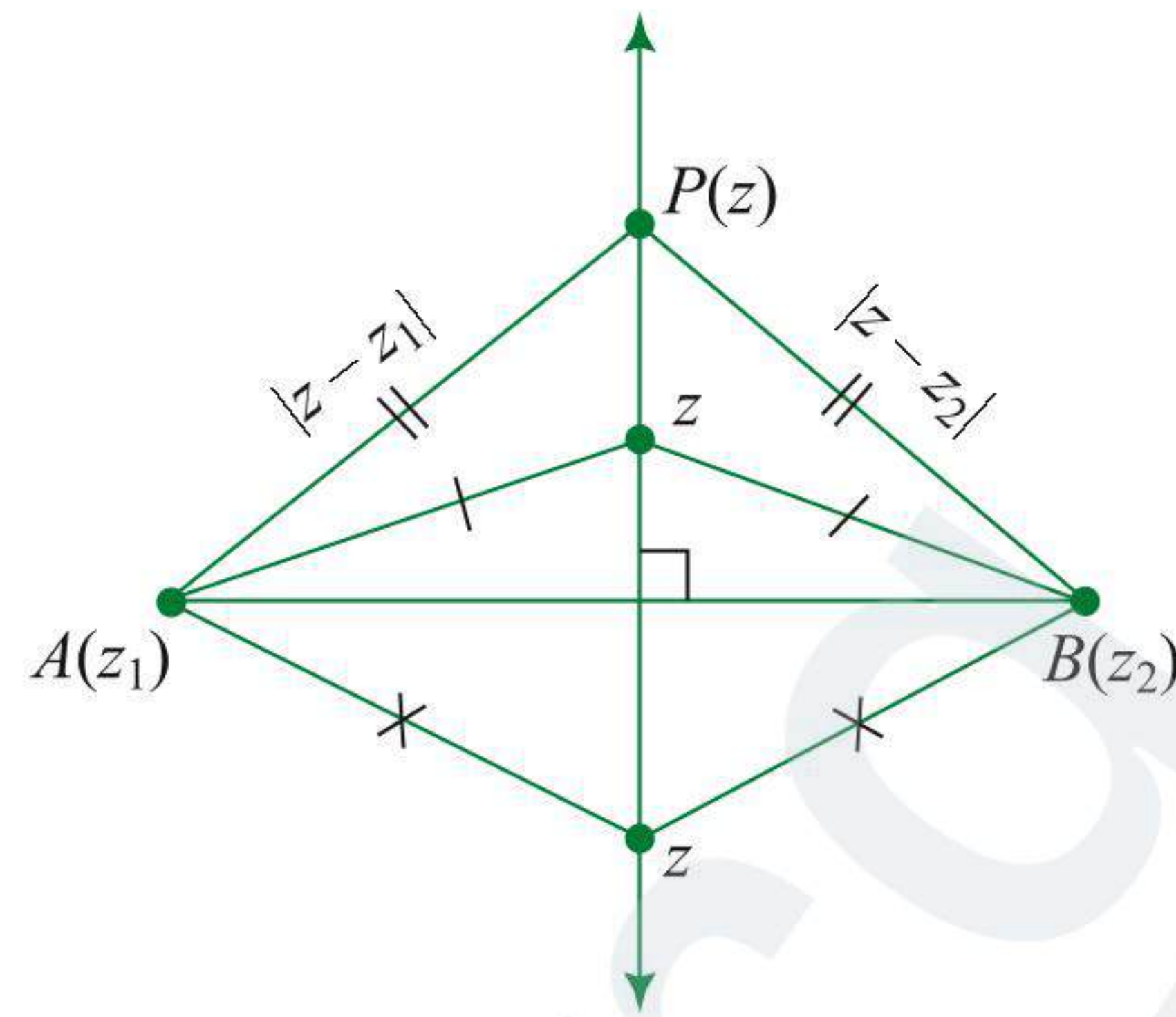
LOCUS BASED ON DISTANCE FORMULA

We know that the distance between two complex numbers z_1 and z_2 in the Argand plane is given by $|z_1 - z_2|$.

Now, let us try to determine the locus of few complex equations using distance formula.

Let $P(z)$ be a variable point and $A(z_1), B(z_2)$ be two fixed points in the Argand plane.

$$1. |z - z_1| = |z - z_2|$$

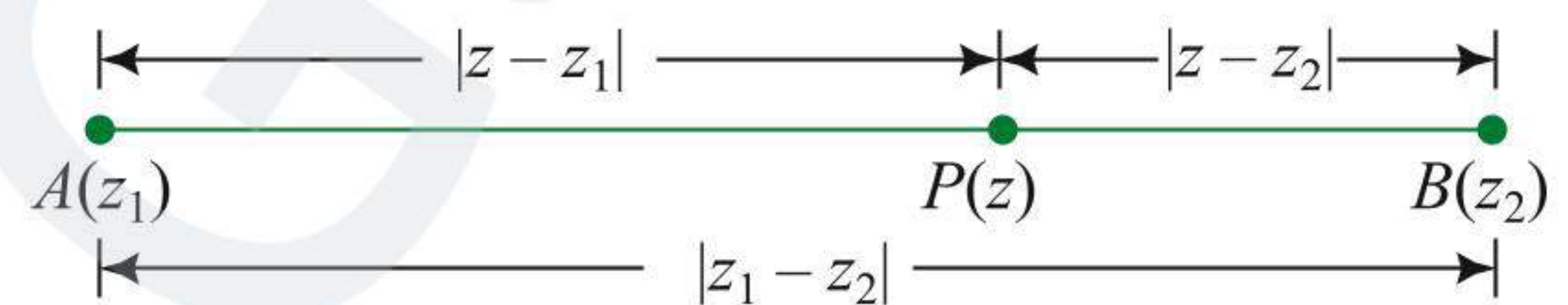


From the figure, $AP = BP$.

Thus, the distance of z from two fixed points z_1 and z_2 is same.

Hence, z lies on the perpendicular bisector of the line segment joining z_1 and z_2 .

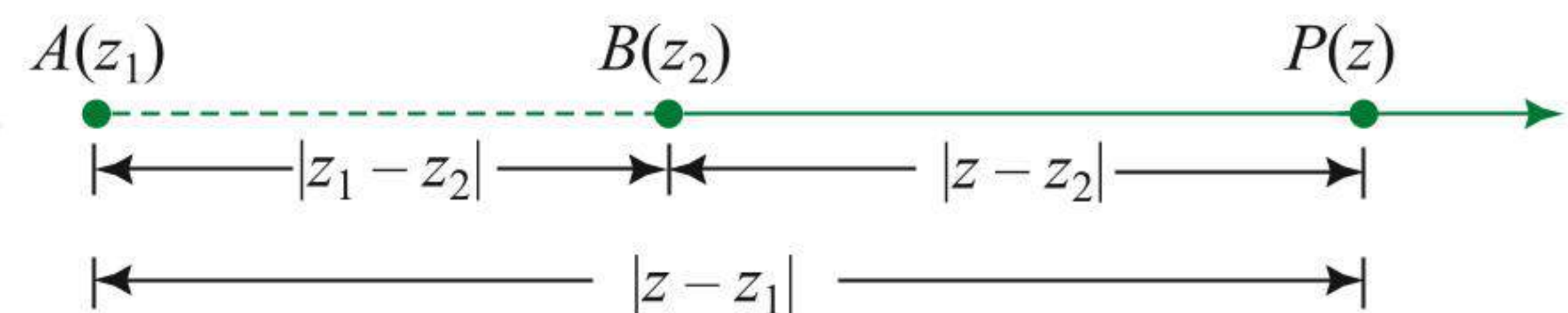
$$2. |z - z_1| + |z - z_2| = |z_1 - z_2|$$



From the figure, $AP + PB = AB$.

Therefore, z lies on the line segment joining z_1 and z_2 .

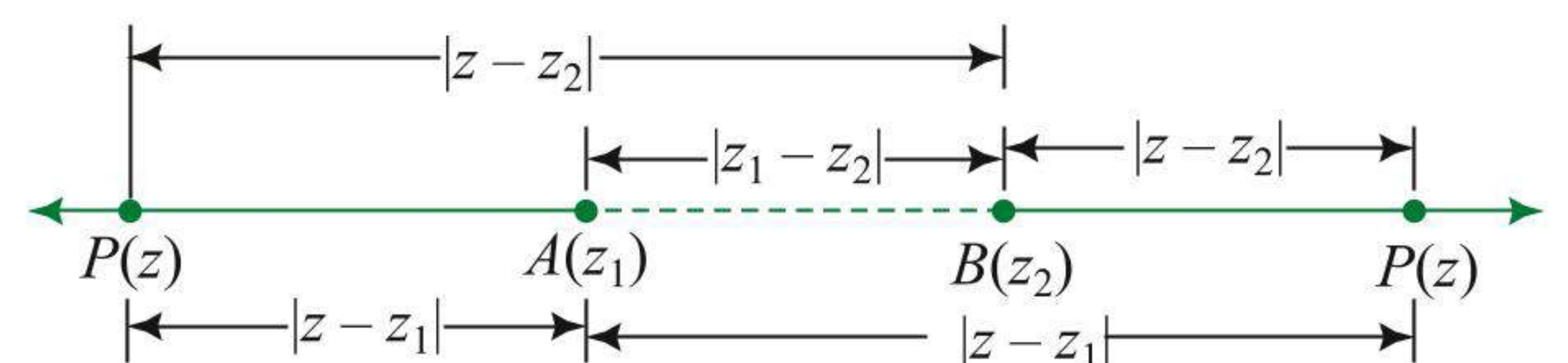
$$3. |z - z_1| - |z - z_2| = |z_1 - z_2|$$



From the figure, $AP - BP = AB$.

Therefore, z lies on the ray emanating from point B as shown in the figure.

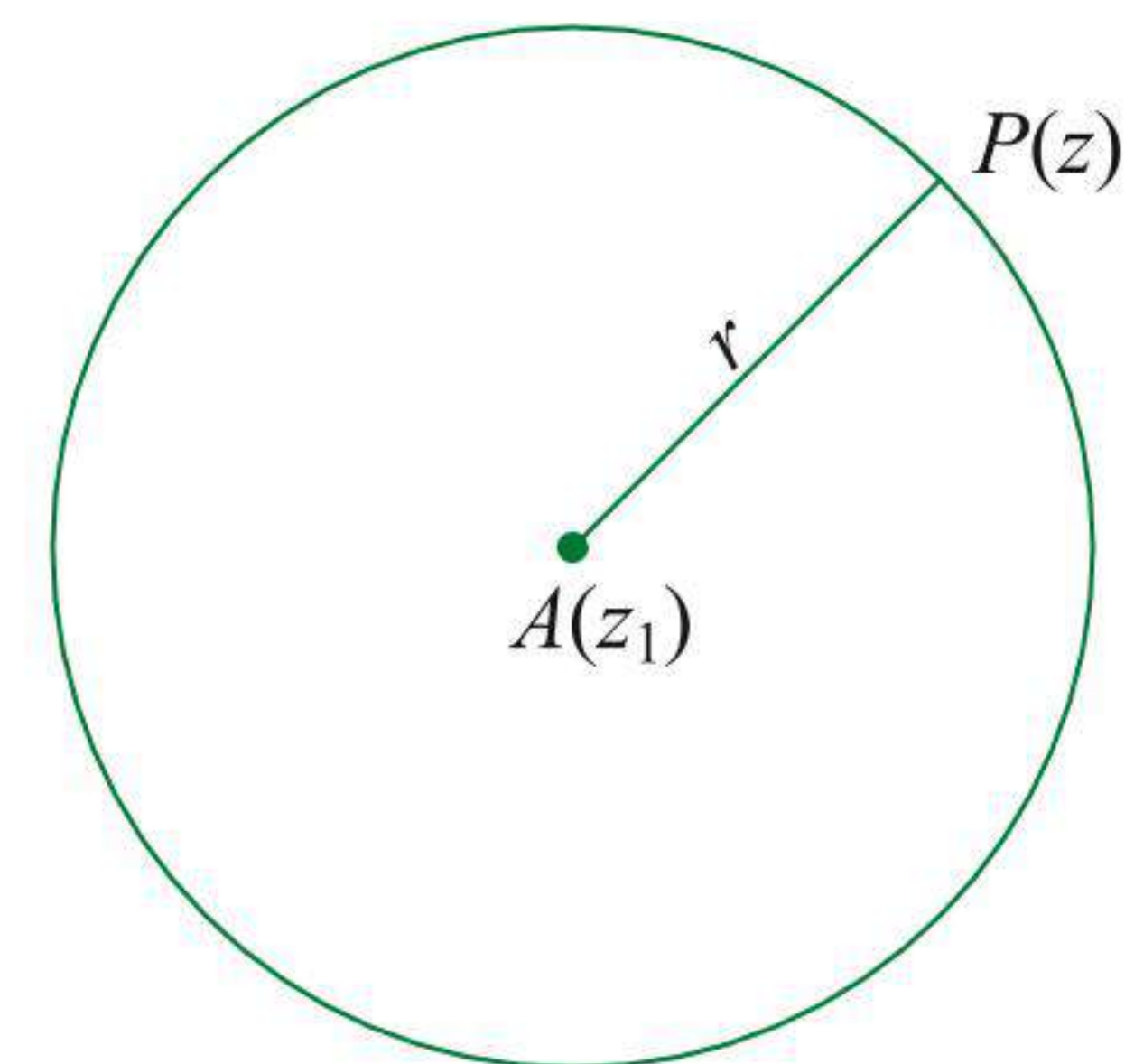
$$4. ||z - z_1| - |z - z_2|| = |z_1 - z_2|$$



From the figure, $|AP - BP| = AB$.

Therefore, z lies on the two rays emanating from points A and B moving in opposite directions.

$$5. |z - z_1| = r; \text{ where } r \text{ is constant.}$$



From the figure, $AP = r$.

i.e., P is at constant distance from A .

So, z lies on the circle whose centre is at z_1 and radius is r .

Also, from $|z - z_1| = r$

$$|z - z_1|^2 = r^2$$

$$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_1) = r^2$$

$$\Rightarrow z\bar{z} - z\bar{z}_1 - \bar{z}z_1 + z_1\bar{z}_1 - r^2 = 0$$

Let $-a = z_1$ and $z_1\bar{z}_1 - r^2 = b$, where $b \in \mathbb{R}$.

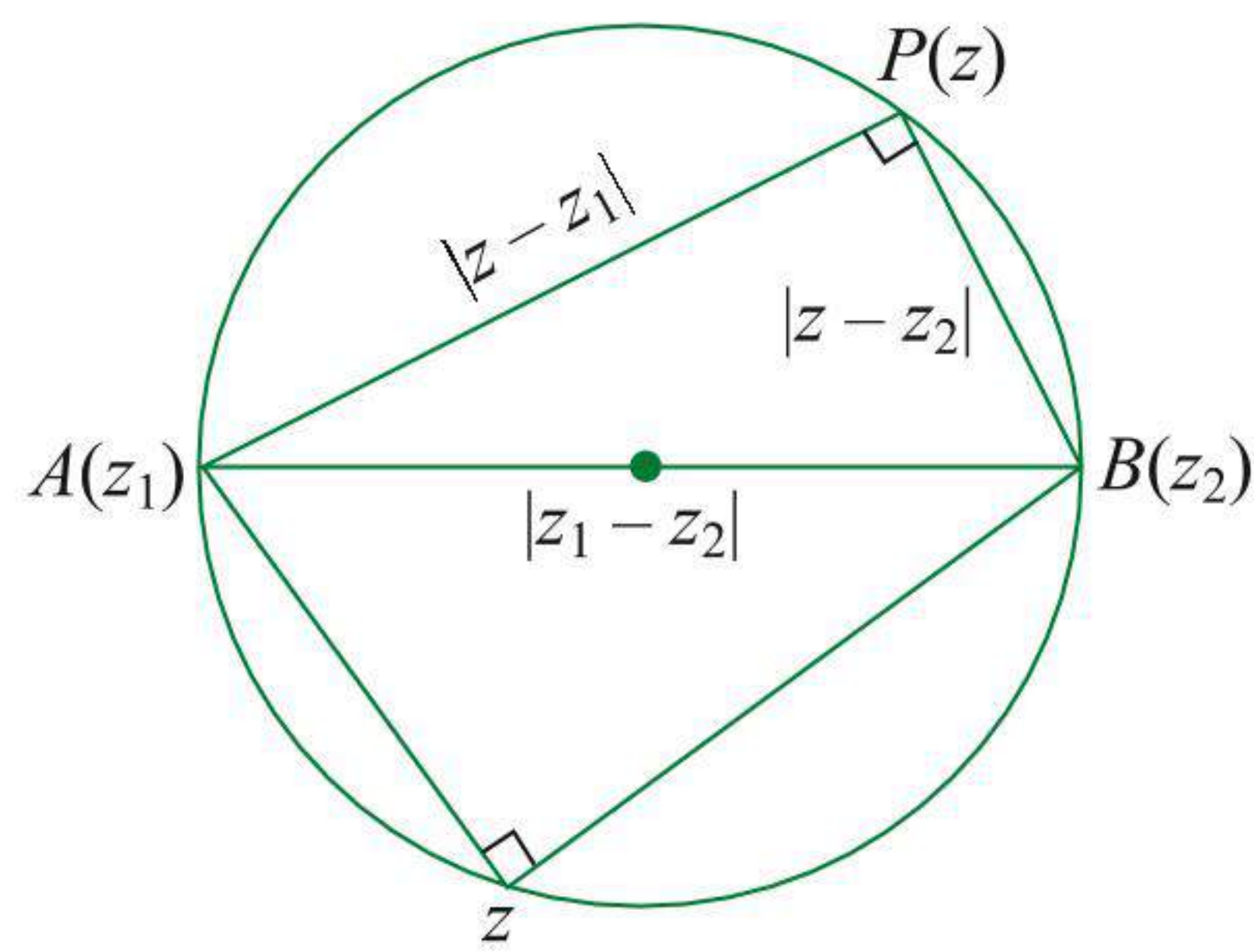
Therefore, equation of circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$.

Centre of circle is $-a$ and radius is $\sqrt{a\bar{a} - b}$.

For real circle, $a\bar{a} - b \geq 0$.

Points satisfying the equation $|z - z_1| \leq r$, lies either on the circle or inside the circle whose centre is z_1 and radius is r .

6. $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

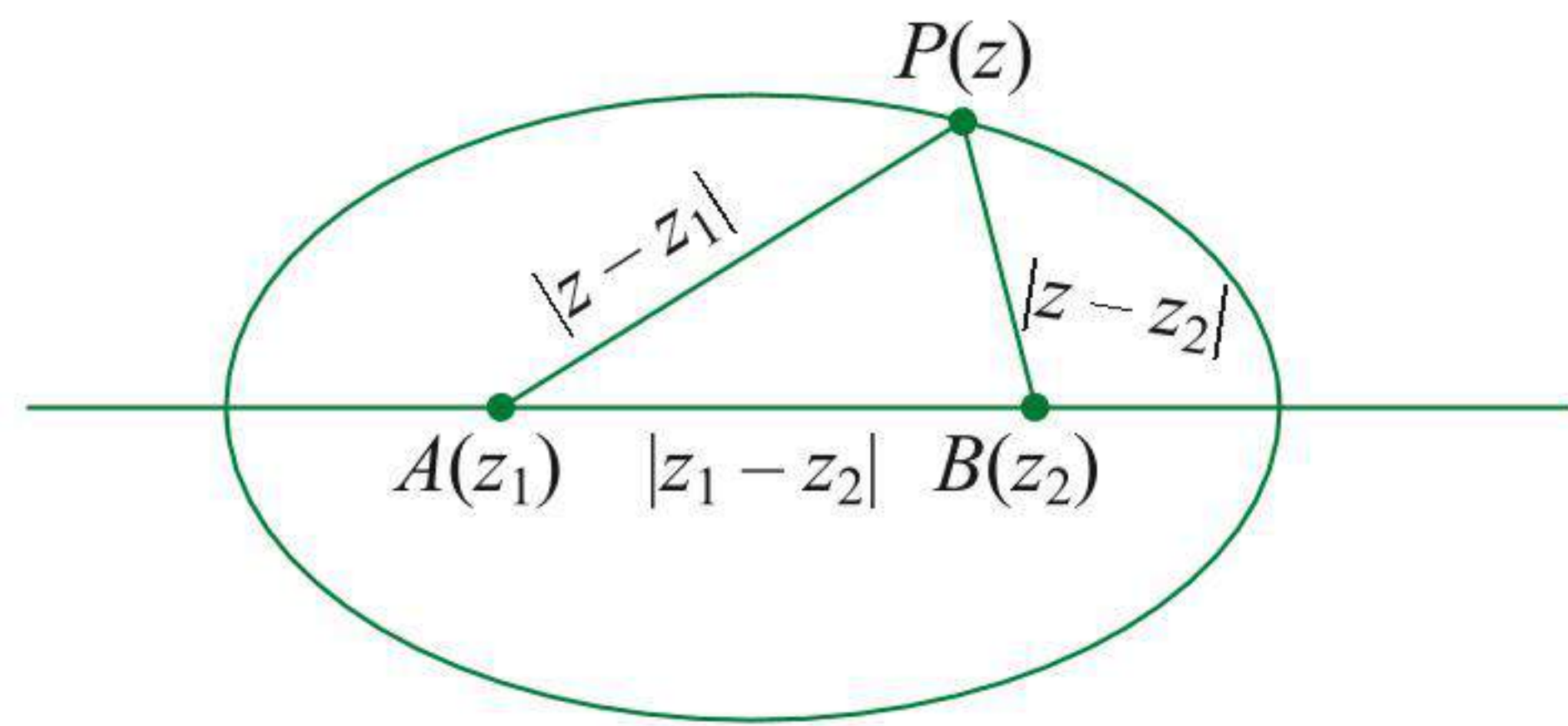


From the figure, $AP^2 + BP^2 = AB^2$.

So, angle subtended by AB at point P is 90° .

Hence, z lies on the circle whose diameter is line segment joining z_1 and z_2 .

7. $|z - z_1| + |z - z_2| = k$; where k is constant and $k > |z_1 - z_2|$.



From the figure, $AP + BP = k$ (constant).

$$\Rightarrow AP + PB + AB = k + AB \text{ (constant)}$$

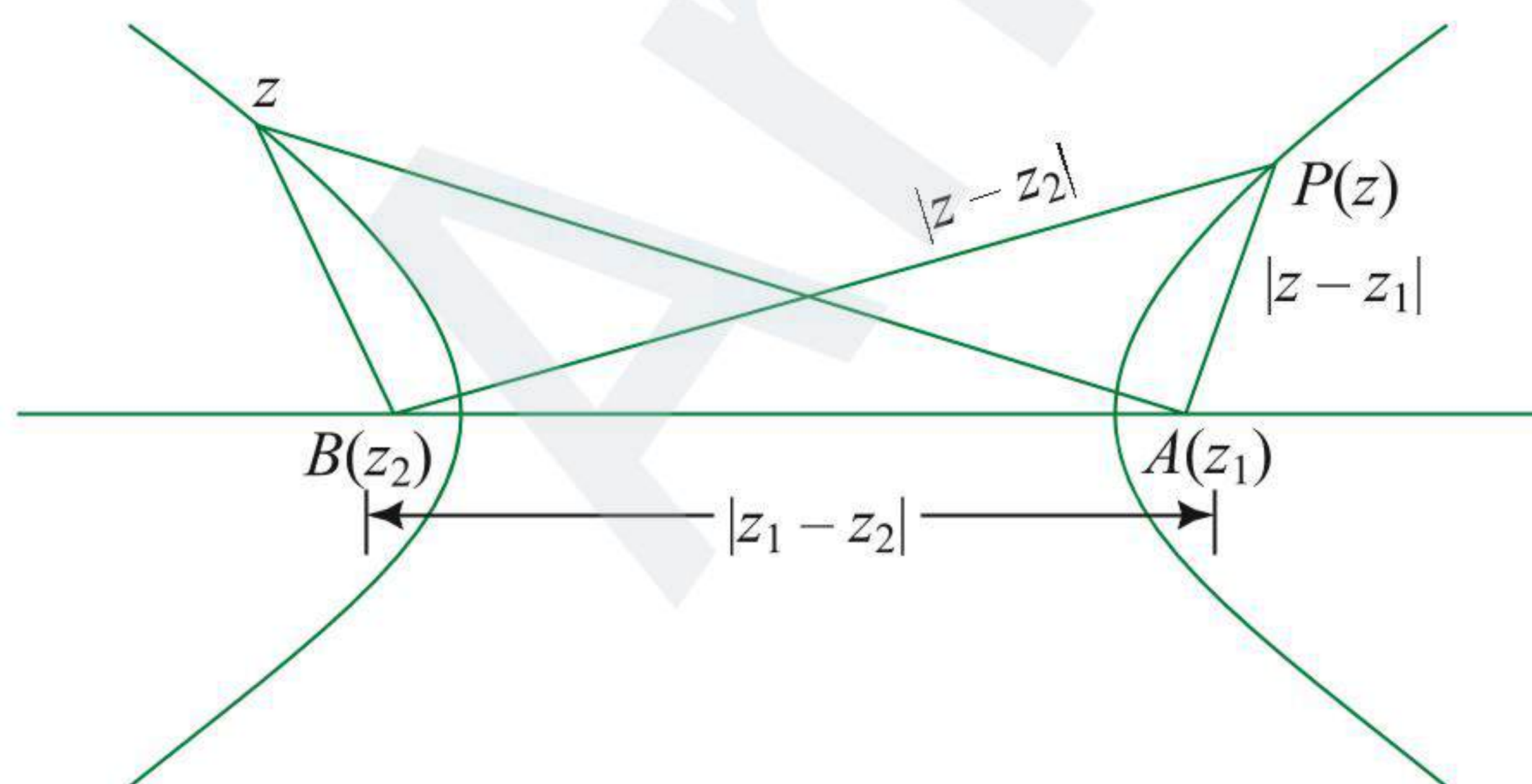
So, perimeter of triangle ABP is constant.

So, z lies on the ellipse whose foci are z_1 and z_2 .

Also, $|z_1 - z_2| = 2ae$, where a is semi-major axis and e is eccentricity.

$$\therefore \text{Eccentricity of ellipse, } e = \frac{|z_1 - z_2|}{k}$$

8. $||z - z_1| - |z - z_2|| = k$; where k is constant and $k < |z_1 - z_2|$.



$$\therefore |AP - BP| = k \text{ (constant)}$$

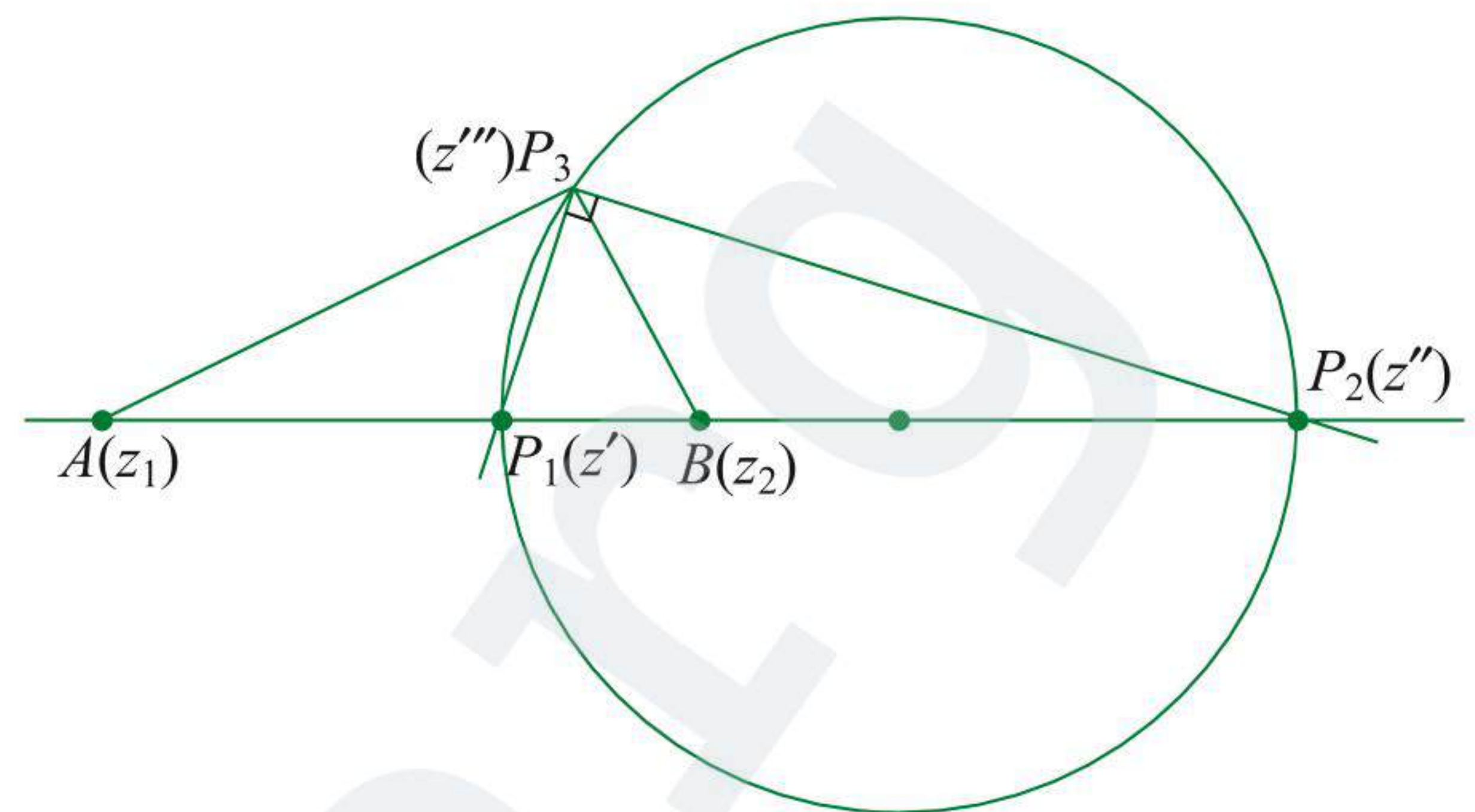
So, z lies on the hyperbola whose foci are z_1 and z_2 .

Also, $|z_1 - z_2| = 2ae$, where a is semi-major axis and e is eccentricity.

$$\therefore \text{Eccentricity of hyperbola, } e = \frac{|z_1 - z_2|}{k}$$

9. $\frac{|z - z_1|}{|z - z_2|} = k, k \neq 0, 1$

$$\text{We have } \frac{AP}{BP} = k$$



We have two positions of point $P(z)$ as $P_1(z')$ and $P_2(z'')$. In the ratio $k : 1$, P_1 and P_2 divide AB internally and externally, respectively.

Now, we have another position of P in the plane as $P_3(z''')$

$$\text{such that } \frac{AP_3}{BP_3} = k.$$

So, from the property of angle bisector, in triangle AP_3B , P_3P_1 is angle bisector of $\angle AP_3B$.

Also, P_3P_2 is external angle bisector of $\angle AP_3B$.

Now, $P_3P_1 \perp P_3P_2$ (as two bisectors are always perpendicular).

So, points P_1, P_2 and P_3 lie on the circle having P_1P_2 as a diameter.

Thus, z lies on the circle.

If $k > 1$, z_2 lies inside the circle.

If $k < 1$, z_1 lies inside the circle.

$$\text{Diameter of the circle} = P_1P_2 = P_1B + BP_2$$

$$= \frac{|z_1 - z_2|}{k+1} + \frac{|z_1 - z_2|}{k-1} = \frac{2k|z_1 - z_2|}{k^2 - 1}, (k > 1)$$

$$\therefore \text{Radius} = \frac{k|z_1 - z_2|}{k^2 - 1}$$

$$\text{Also, } z' = \frac{kz_2 + z_1}{k+1} \text{ and } z'' = \frac{kz_2 - z_1}{k-1}.$$

$$\text{Thus, centre is mid-point of } P_1P_2 \text{ given by } \frac{k^2 z_2 - z_1}{k^2 - 1}.$$

ILLUSTRATION 3.77

Identify the locus of z if $\bar{z} = \bar{a} + \frac{r^2}{z - a}$, $r > 0$.

Sol. $\bar{z} = \bar{a} + \frac{r^2}{z - a}$

or $\bar{z} - \bar{a} = \frac{r^2}{z - a}$

or $(z - a)(\bar{z} - \bar{a}) = r^2$

or $|z - a|^2 = r^2$

or $|z - a| = r$

Hence, locus of z is circle having center a and radius r .

ILLUSTRATION 3.78

If z is any complex number such that $|3z - 2| + |3z + 2| = 4$, then identify the locus of z .

Sol. $|3z - 2| + |3z + 2| = 4$

$$\Rightarrow \left| z - \frac{2}{3} \right| + \left| z + \frac{2}{3} \right| = \frac{4}{3} \quad (1)$$

If $P(z)$ be any point $A \equiv (2/3, 0)$, $B \equiv (-2/3, 0)$, then (1) represents

$$PA + PB = 4$$

Clearly, $AB = 4/3 \Rightarrow PA + PB = AB$

$\Rightarrow P$ lies on the line segment AB .

ILLUSTRATION 3.79

If $|z| = 1$ and let $\omega = \frac{(1-z)^2}{1-z^2}$, then prove that the locus of ω is equivalent to $|z - 2| = |z + 2|$.

Sol. Given

$$\omega = \frac{(1-z)^2}{1-z^2} = \frac{1-z}{1+z}$$

$$\Rightarrow \omega = \frac{z\bar{z} - z}{z\bar{z} + z} = \frac{\bar{z} - 1}{\bar{z} + 1} = -\left(\frac{1-\bar{z}}{1+z}\right) = -\bar{\omega}$$

$$\therefore \omega + \bar{\omega} = 0$$

$\Rightarrow \omega$ is purely imaginary. Hence, ω lies on the y -axis.

Also $|z - 2| = |z + 2| \Rightarrow z$ lies on perpendicular bisector of line segment joining 2 and -2 , which is the imaginary axis.

ILLUSTRATION 3.80

Let z be a complex number having the argument θ , $0 < \theta < \pi/2$, and satisfying the equation $|z - 3i| = 3$. Then find the value of $\cot \theta - 6/z$.

Sol. $|z - 3i| = 3$

So, z lies on circle having centre $3i$ and radius 3.

Let $z = r(\cos \theta + i \sin \theta)$.

Now, $r = OA \sin \theta = 6 \sin \theta$

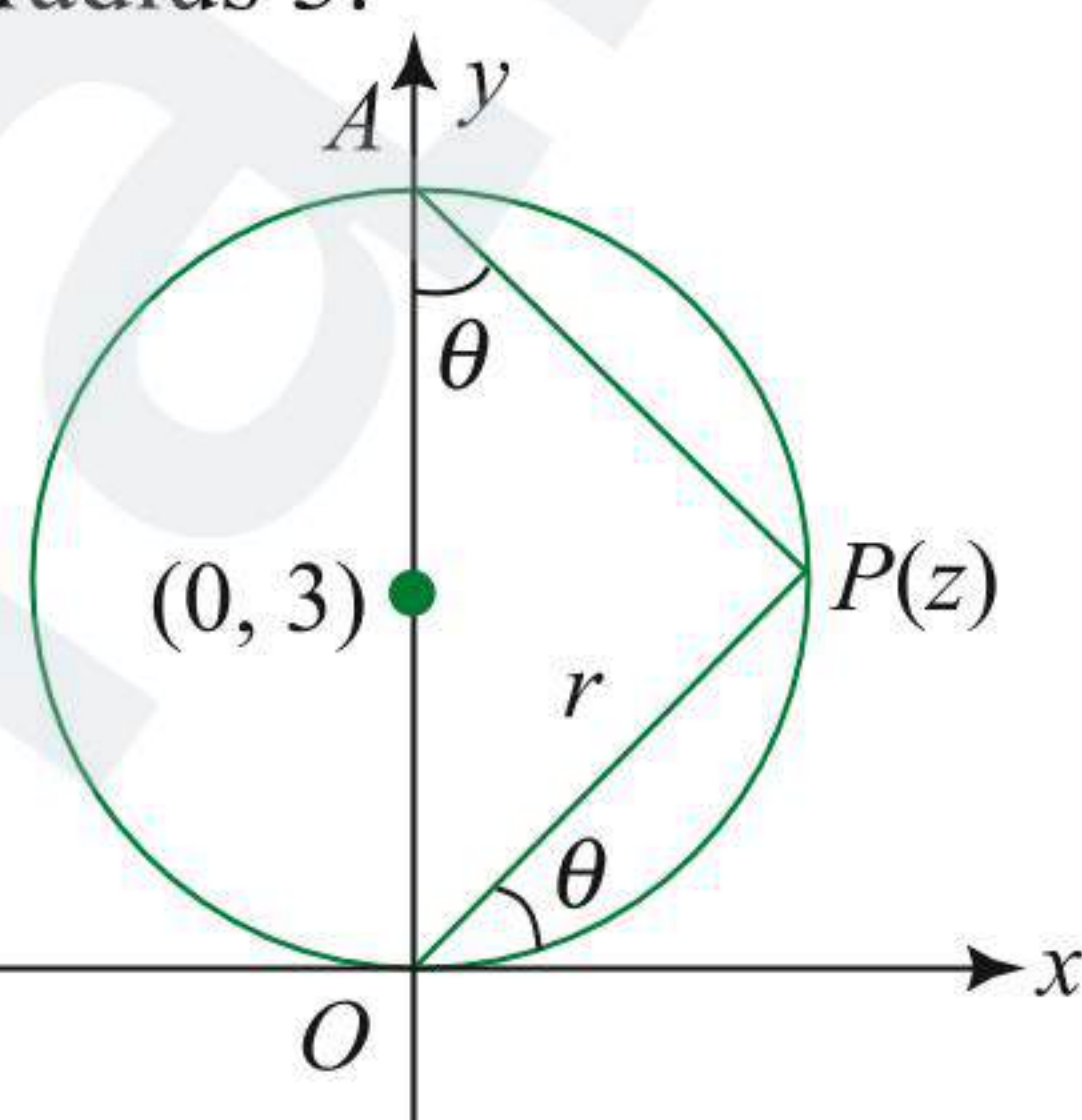
$$\Rightarrow z = 6 \sin \theta (\cos \theta + i \sin \theta)$$

$$\text{or } \frac{6}{z} = \frac{1}{\sin \theta (\cos \theta + i \sin \theta)}$$

$$= \frac{\cos \theta - i \sin \theta}{\sin \theta}$$

$$= -i + \cot \theta$$

$$\text{or } \cot \theta - \frac{6}{z} = i$$

**ILLUSTRATION 3.81**

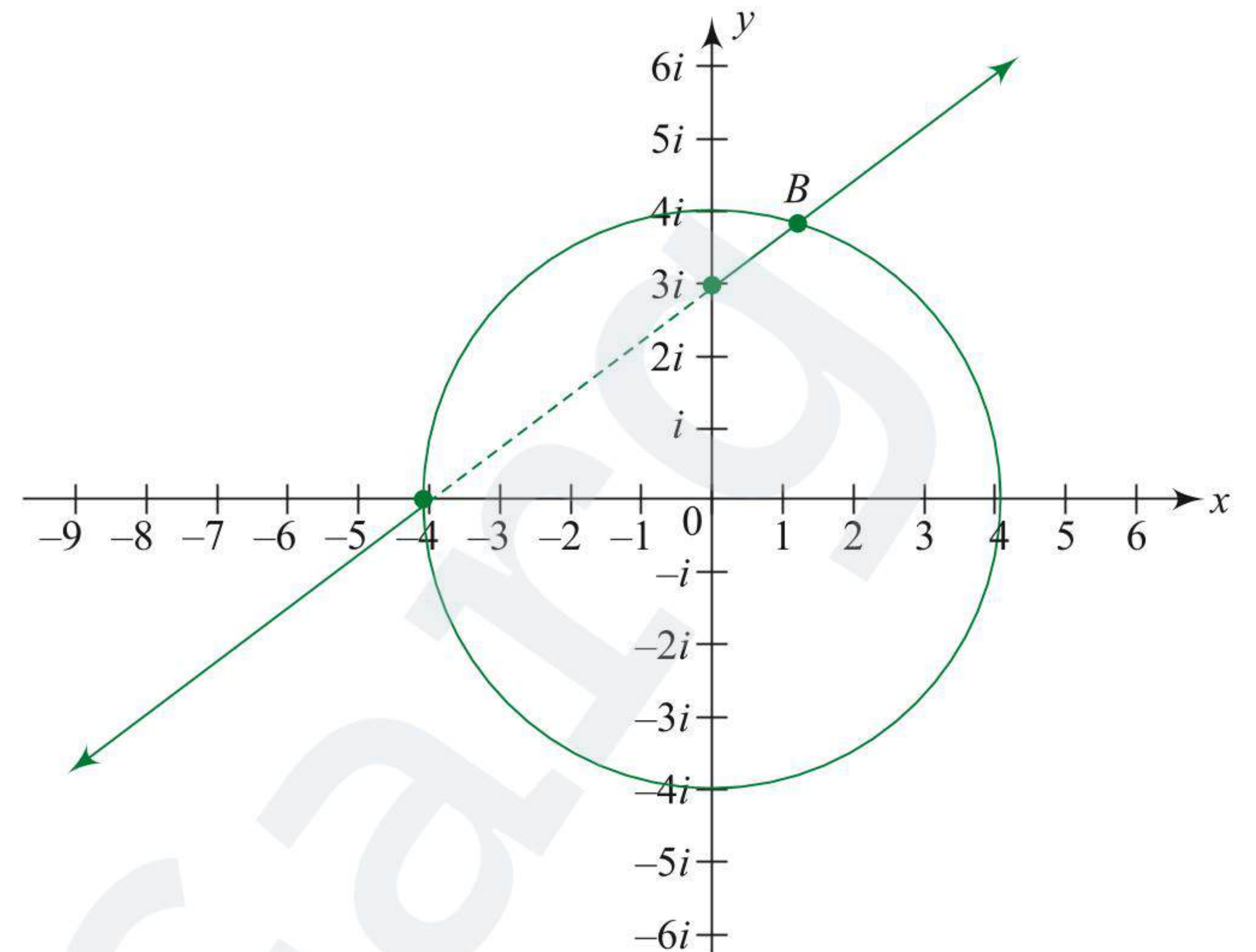
How many solutions the system of equations $||z + 4| - |z - 3i|| = 5$ and $|z| = 4$ has?

Sol. $||z + 4| - |z - 3i|| = 5$

$$\text{or } ||z + 4| - |z - 3i|| = |4 + 3i|$$

This represents two rays emanating from -4 and $3i$ and moving in opposite directions.

$|z| = 4$ represents a circle having centre at origin and radius '4'.



Clearly, curves intersect at two points, A and B .

Hence, system of equations has two solutions.

ILLUSTRATION 3.82

Prove that $|z - z_1|^2 + |z - z_2|^2 = a$ will represent a real circle on the Argand plane if $2a \geq |z_1 - z_2|^2$.

Sol. $|z - z_1|^2 + |z - z_2|^2 = a$

$$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_1) + (z - z_2)(\bar{z} - \bar{z}_2) = a$$

$$\Rightarrow 2z\bar{z} - z(\bar{z}_1 + \bar{z}_2) - \bar{z}(z_1 + z_2) + z_1\bar{z}_1 + z_2\bar{z}_2 = a$$

$$\Rightarrow z\bar{z} - z\left(\frac{\bar{z}_1 + \bar{z}_2}{2}\right) - \bar{z}\left(\frac{z_1 + z_2}{2}\right) + \frac{z_1\bar{z}_1 + z_2\bar{z}_2 - a}{2} = 0 \quad (1)$$

Above equation is of the form $z\bar{z} + \alpha\bar{z} + \bar{\alpha}z + \beta = 0$, which represents the real circle if $\alpha\bar{\alpha} - \beta \geq 0$

$$\Rightarrow \frac{(z_1 + z_2)(\bar{z}_1 + \bar{z}_2)}{4} \geq \frac{z_1\bar{z}_1 + z_2\bar{z}_2 - a}{2}$$

$$\Rightarrow 2a \geq z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_2 - z_2\bar{z}_1$$

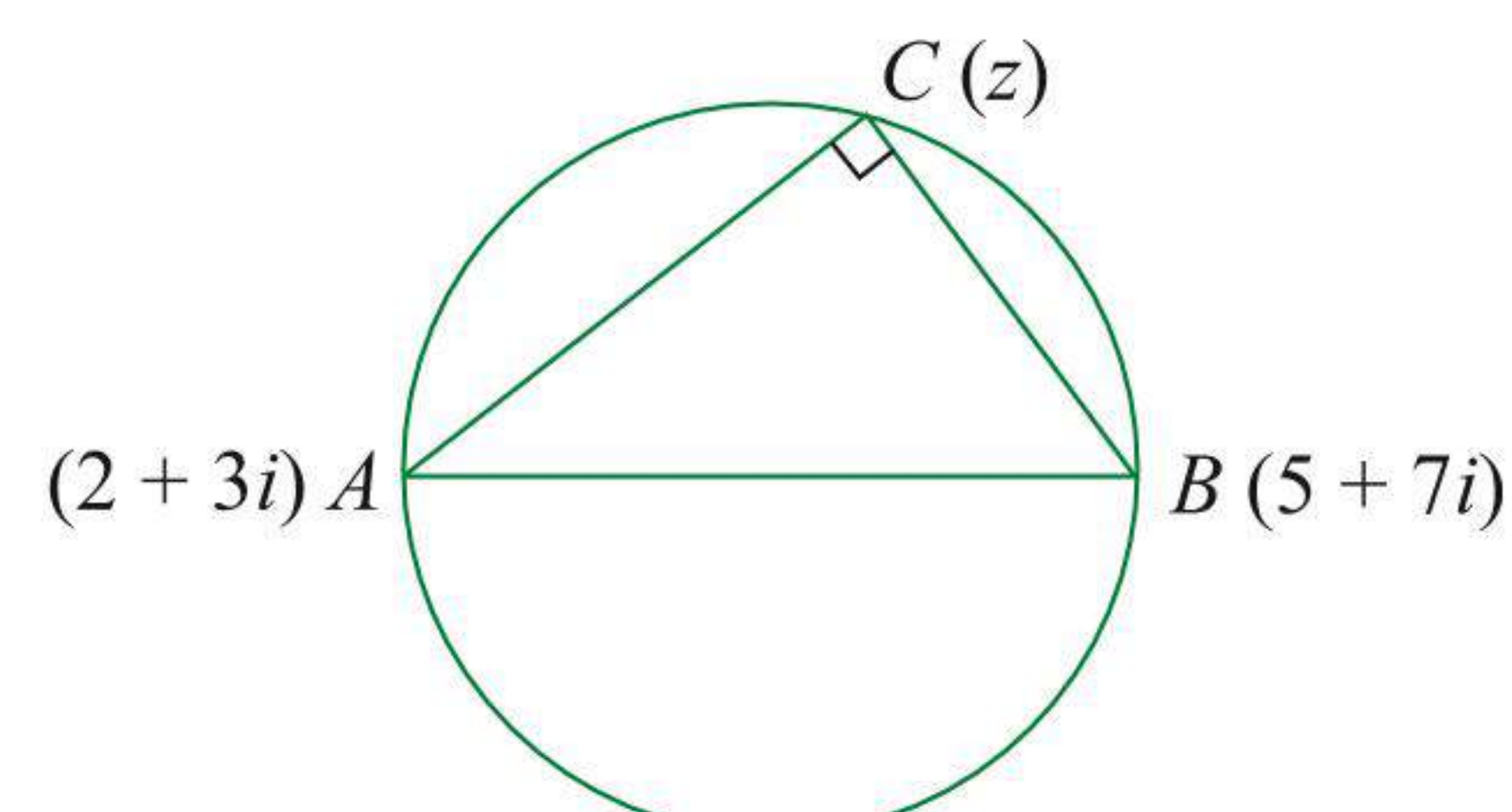
$$\Rightarrow 2a \geq |z_1 - z_2|^2$$

ILLUSTRATION 3.83

If $|z - 2 - 3i|^2 + |z - 5 - 7i|^2 = \lambda$ represents the equation of circle passing through $(2 + 3i)$ and $(5 + 7i)$ with least radius, then find the value of λ .

Sol. We have

$$|z - (2 + 3i)|^2 + |z - (5 + 7i)|^2 = \lambda$$



Clearly, this represents the equation of circle with least radius if $(2 + 3i)$ and $(5 + 7i)$ are end points of diameter of the circle.

$$\therefore \lambda = |(2 + 3i) - (5 + 7i)|^2 = |-3 - 4i|^2 = 5^2 = 25$$

ILLUSTRATION 3.84

If $\frac{|2z-3|}{|z-i|} = k$ is the equation of circle with complex number 'i' lying inside the circle, find the values of k .

Sol. We have

$$\frac{|2z-3|}{|z-i|} = k$$

$$\Rightarrow \frac{|z-3/2|}{|z-i|} = \frac{k}{2}, \text{ this is equation of circle if } k \neq 0, 2$$

If 'i' lies inside the circle then $\frac{k}{2} > 1$ or $k > 2$.

CONCEPT APPLICATION EXERCISE 3.9

1. If $\omega = z/[z - (1/3)i]$ and $|\omega| = 1$, then find the locus of z .
2. If $\operatorname{Im}\left(\frac{z-1}{e^{\theta i}} + \frac{e^{\theta i}}{z-1}\right) = 0$, then find the locus of z .
3. For three non-collinear complex numbers Z, Z_1 and Z_2 , prove that $\left|Z - \frac{Z_1 + Z_2}{2}\right|^2 + \left|\frac{Z_1 - Z_2}{2}\right|^2 = \frac{1}{2}|Z - Z_1|^2 + \frac{1}{2}|Z - Z_2|^2$.
4. If $|z-1| + |z+3| \leq 8$, then find the range of values of $|z-4|$.
5. If $z = \frac{3}{2 + \cos \theta + i \sin \theta}$, then prove that z lies on the circle.
6. How many solutions system of equations, $\arg(z+2-3i) = -\pi/4$ and $|z+4| - |z-3i| = 5$, has?
7. Prove that equation of perpendicular bisector of line segment joining complex numbers z_1 and z_2 is $z(\bar{z}_2 - \bar{z}_1) + \bar{z}(z_2 - z_1) + |z_1|^2 - |z_2|^2 = 0$.
8. If complex number z lies on the curve $|z+1-i| = 1$, then find the locus of the complex number $w = \frac{z+i}{1-i}$, $i = \sqrt{-1}$.

ANSWERS

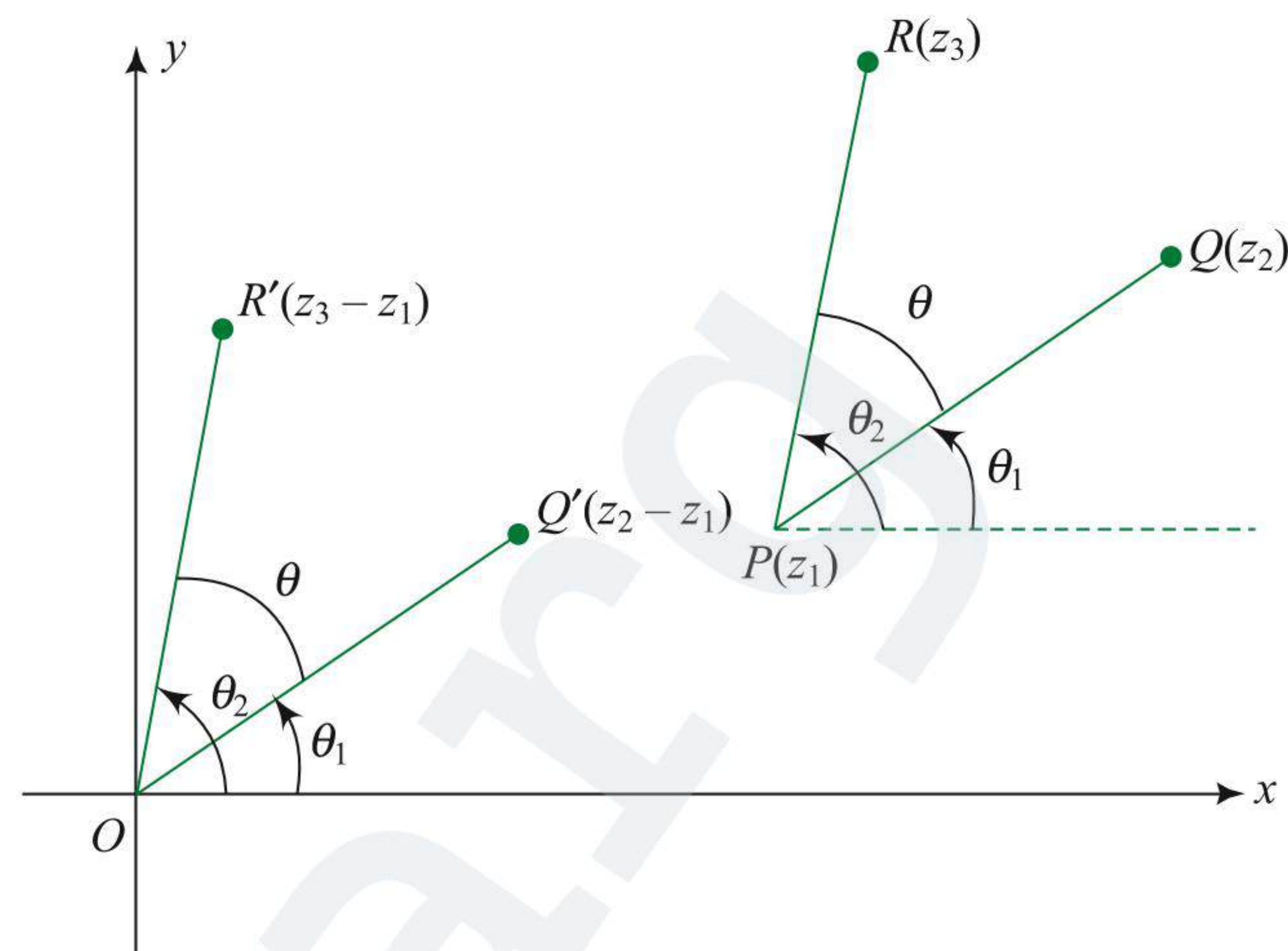
1. Perpendicular bisector of the line joining $0 + 0i$ and $0 + (1/3)i$
2. Circle having center at $1 + i0$ and radius 1
4. $1 \leq |z-4| \leq 9$ 6. No solution
8. A circle with center at $(-3/2, 1/2)$ and radius $1/\sqrt{2}$

ANGLE BETWEEN TWO LINES AND ROTATION FORMULA**ANGLE BETWEEN TWO RAYS**

We know that in the Argand plane, direction of ray is measured in terms of argument.

Now, let us get the formula for the angle between two rays.

Consider three points $P(z_1)$, $Q(z_2)$ and $R(z_3)$ in the argand plane as shown in the figure.



Let angle subtended by QR at point P be θ .

Now, shift $\angle RPQ$ at origin so that points P, Q and R shift to points O, Q' and R' , respectively.

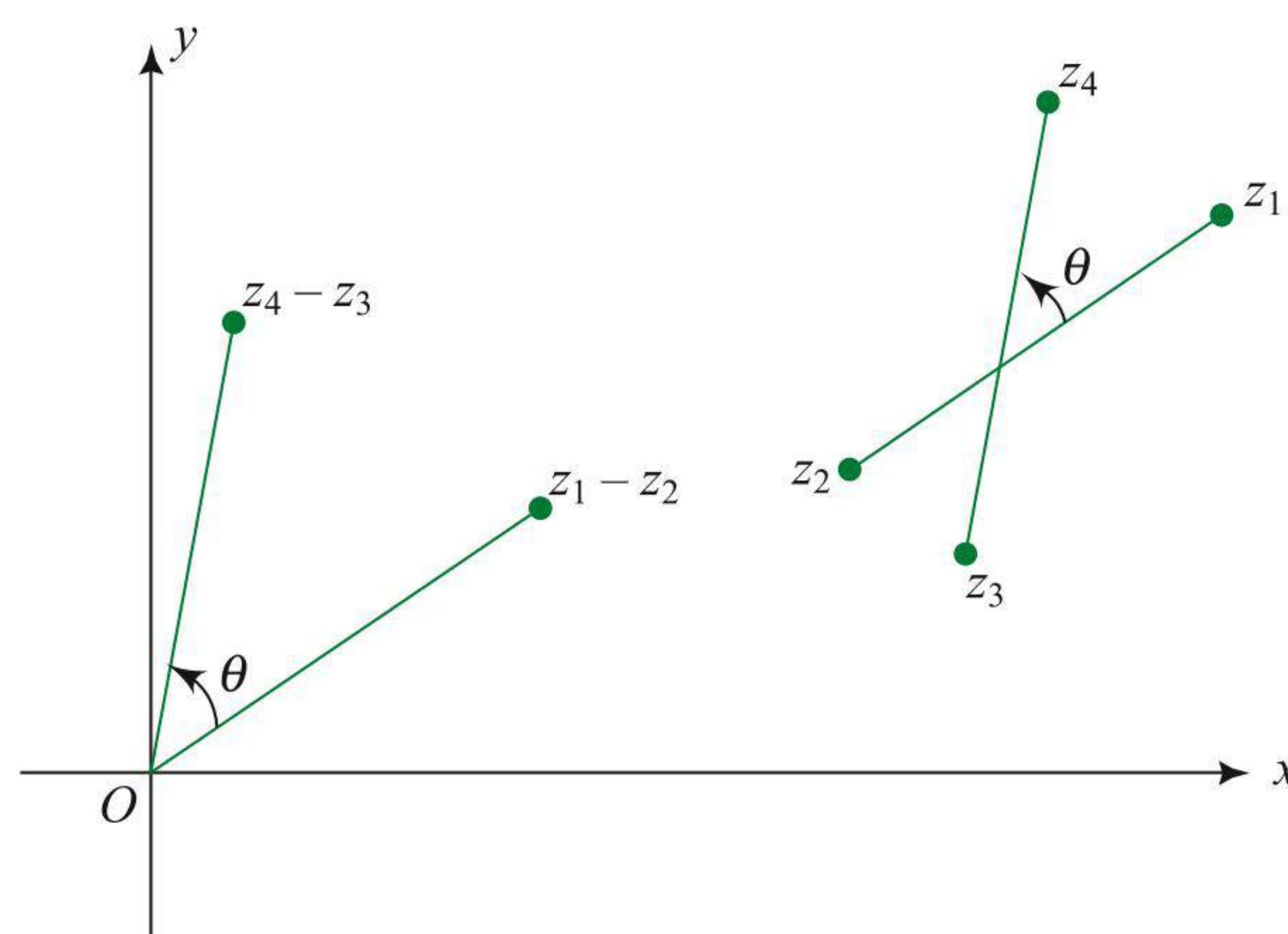
So, $OQ' = PQ = |z_2 - z_1|$ and $OR' = PR = |z_3 - z_1|$

Also, OQ' is parallel to PQ and OR' is parallel to PR .

So, complex number for point Q' is $(z_2 - z_1)$ and that for point R' is $(z_3 - z_1)$ such that $\arg(z_2 - z_1) = \theta_1$ and $\arg(z_3 - z_1) = \theta_2$.

$$\therefore \theta = \theta_2 - \theta_1 = \arg(z_3 - z_1) - \arg(z_2 - z_1) = \arg \frac{z_3 - z_1}{z_2 - z_1}$$

Also, angle between two lines joining z_1 to z_2 and z_3 to z_4 is $\arg \frac{z_4 - z_3}{z_1 - z_2}$.

**EQUATION OF RAY**

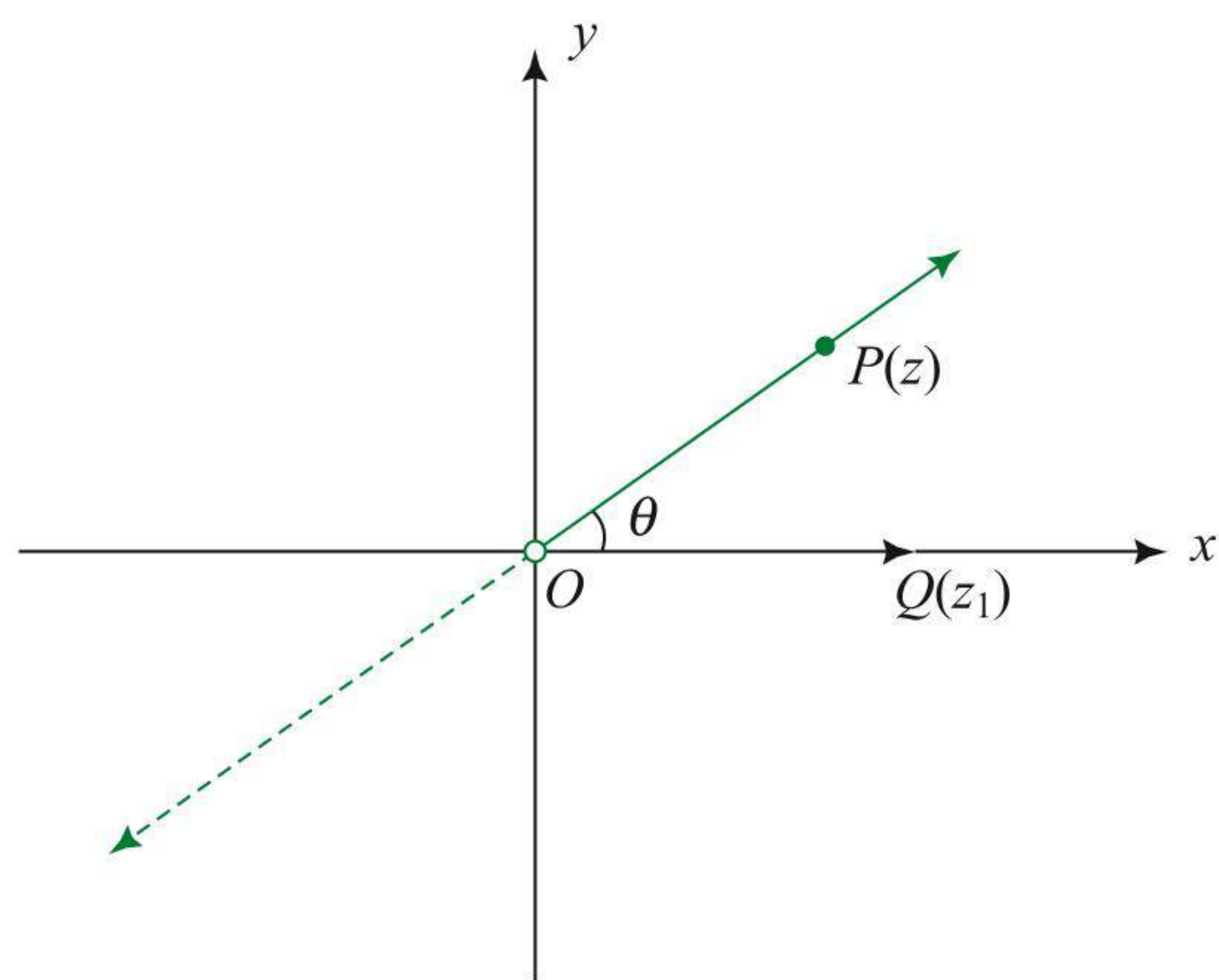
We know that infinite complex numbers with the same argument θ lie on the ray emanating from the origin but excluding origin.

Now, $\arg(z) = \theta$, where $z = x + iy$

$$\therefore \tan^{-1} \frac{y}{x} = \theta$$

$$\therefore \frac{y}{x} = \tan \theta$$

$$\text{or } y = (\tan \theta)x$$



So, this ray is the part of the line passing through origin whose equation is $y = (\tan \theta)x$. However, the direction of the ray is decided by the argument θ .

The equation of ray can also be obtained using angle formula. In above figure, $\angle POQ = \theta$.

So, equation of ray is $\arg \frac{z-0}{z_1-0} = \theta$, where z_1 is any complex

number on positive real axis.

$$\therefore \arg z - \arg z_1 = \theta$$

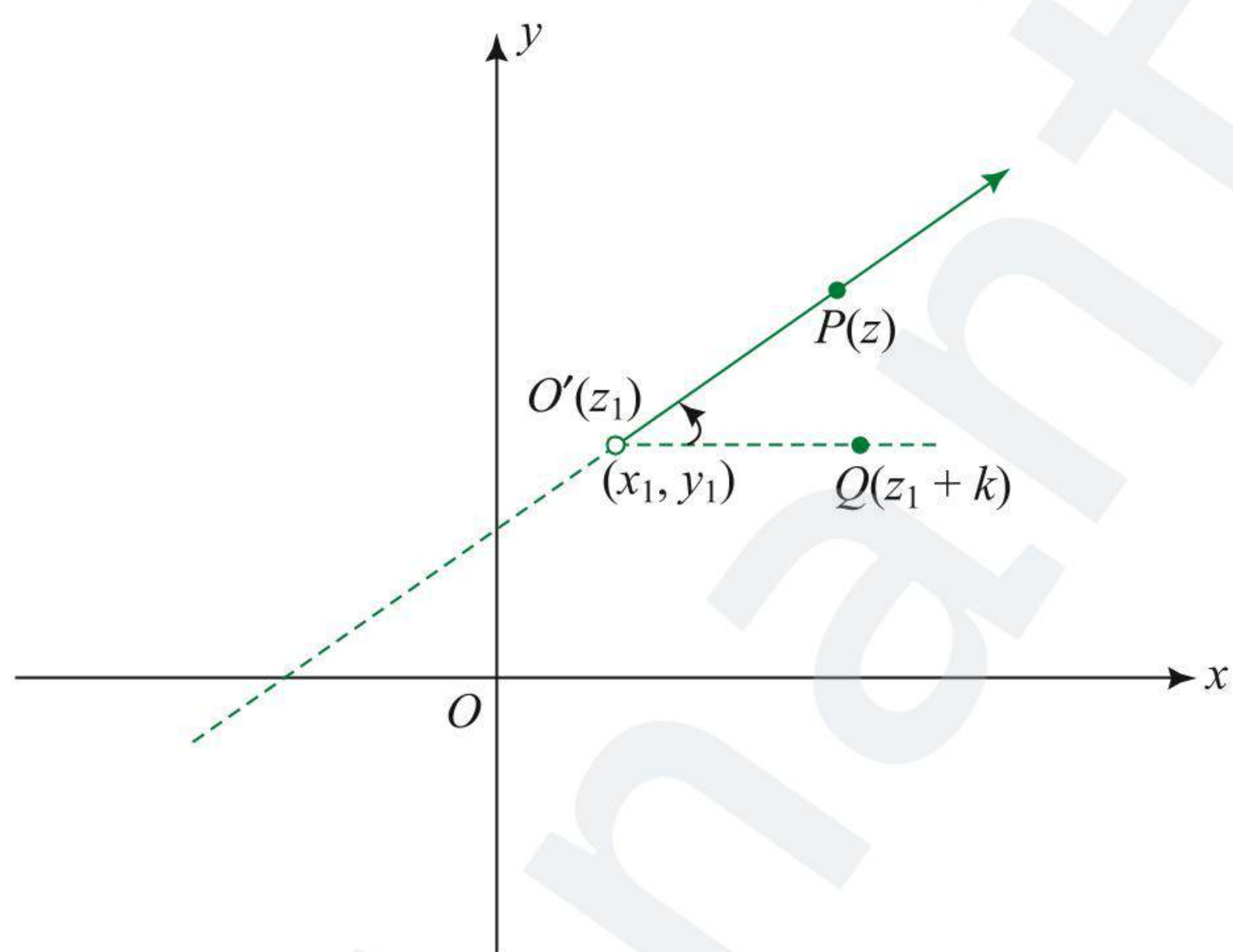
$$\Rightarrow \arg z - 0 = \theta$$

$$\Rightarrow \arg z = \theta$$

Now, consider equation $\arg(z - z_1) = \theta$. Points satisfying this equation lie on the ray emanating from z_1 (excluding z_1).

This ray is the part of line through (x_1, y_1) and slope $\tan \theta$ whose equation is $y - y_1 = (\tan \theta)(x - x_1)$.

The direction of the ray is according to the argument θ .



Let us get the equation of ray using angle formula.

In above figure, $\angle PO'Q = \theta$

So, equation of ray is $\arg \frac{z - z_1}{(z_1 + k) - z_1} = \theta$, where $z_1 + k$ ($k \in \mathbb{R}$)

is any complex number on the horizontal through z_1 lying on right hand side.

$$\therefore \arg \frac{z - z_1}{k} = \theta$$

$$\Rightarrow \arg(z - z_1) - \arg k = \theta$$

$$\Rightarrow \arg(z - z_1) - 0 = \theta$$

$$\Rightarrow \arg(z - z_1) = \theta$$

Some of the rays are plotted as shown in the following figure.

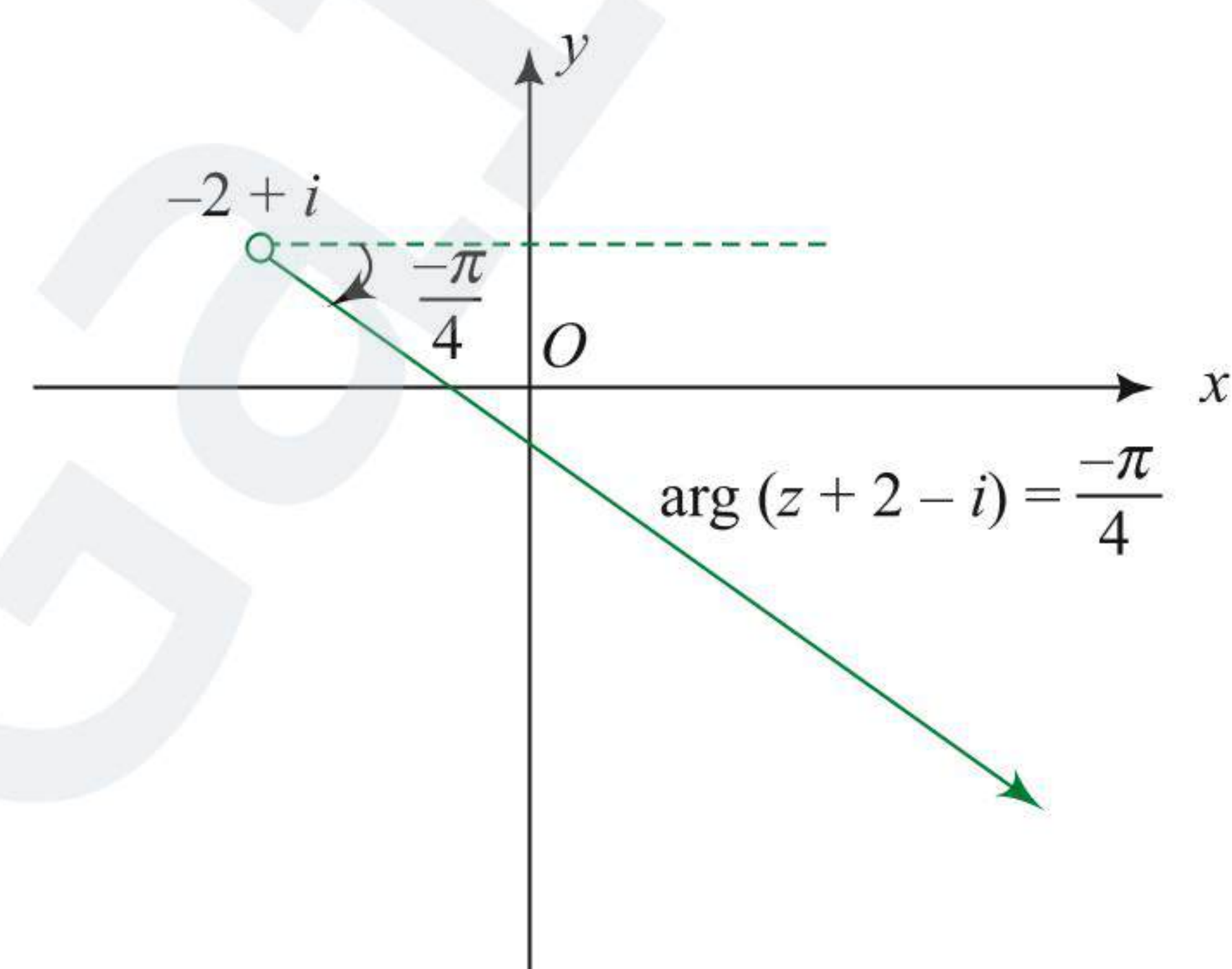
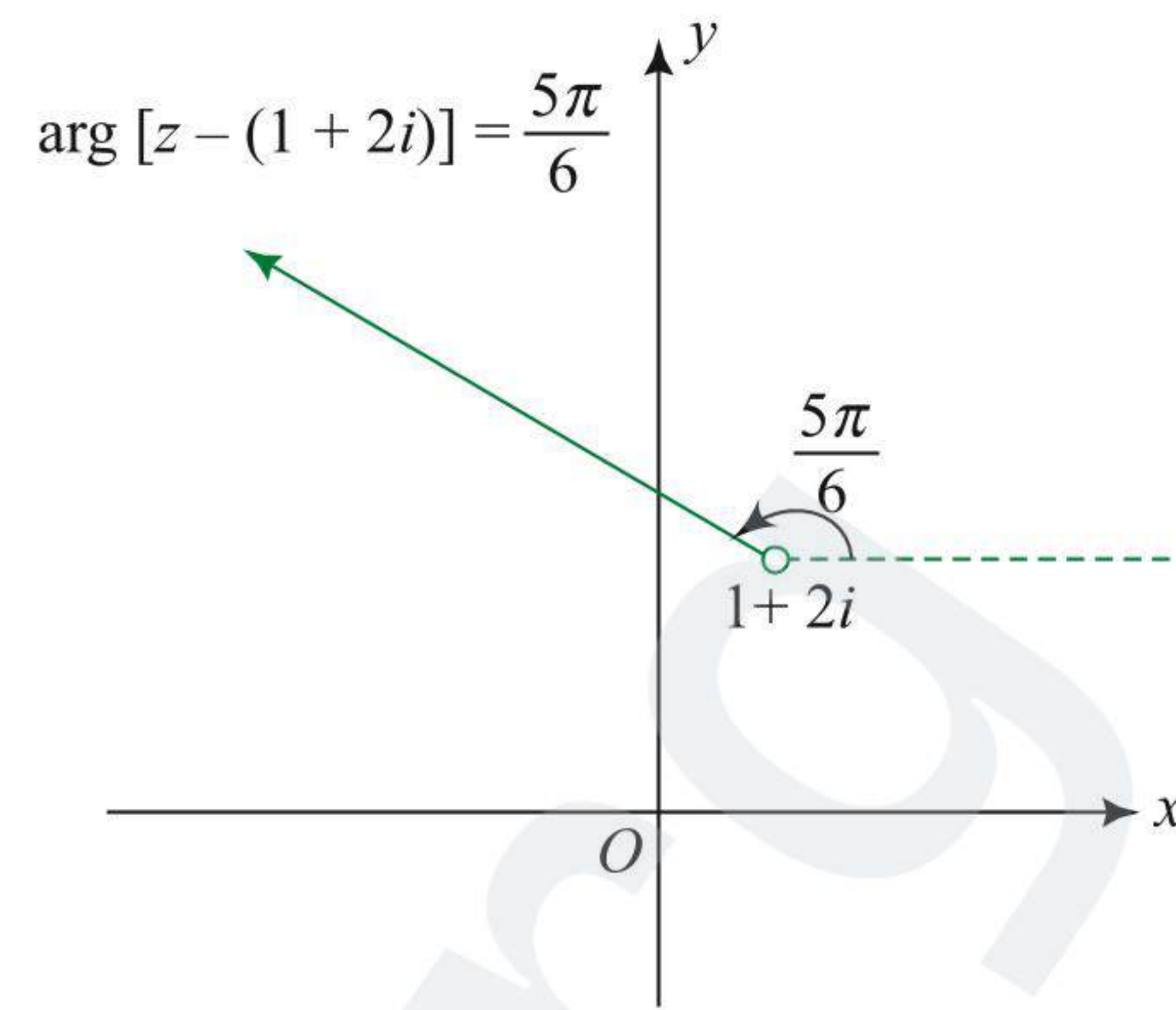
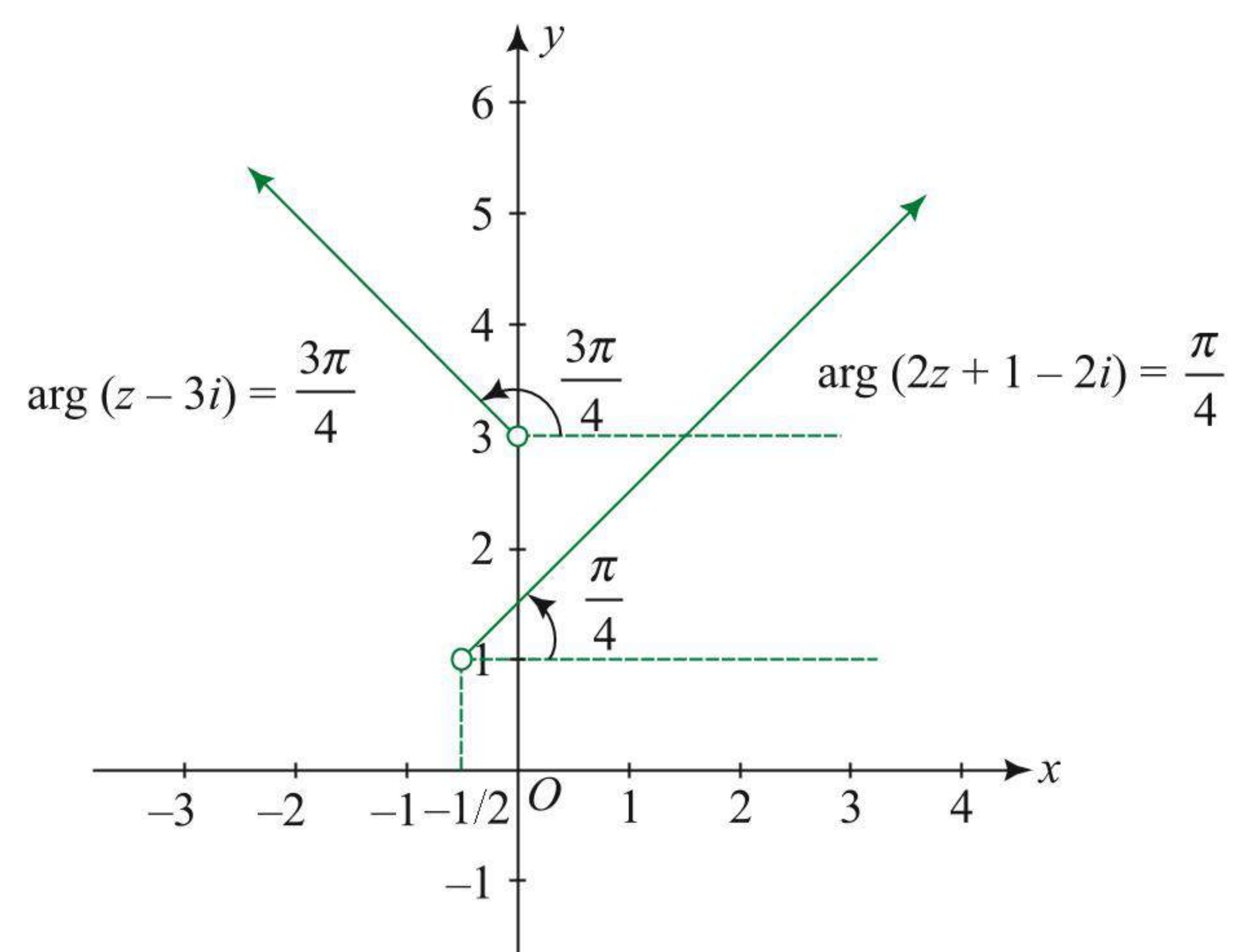


ILLUSTRATION 3.85

Find the point of intersection of the curves $\arg(z - 3i) = 3\pi/4$ and $\arg(2z + 1 - 2i) = \pi/4$.

Sol. We have, $\arg(z - 3i) = \frac{3\pi}{4}$

which is a ray that starts from $3i$ and makes an angle $3\pi/4$ with the positive real axis as shown in the figure.



$$\arg(2z + 1 - 2i) = \frac{\pi}{4}$$

$$\text{or } \arg \left[2 \left(z + \frac{1}{2} - i \right) \right] = \frac{\pi}{4}$$

$$\text{or } \arg 2 + \arg \left[z - \left(-\frac{1}{2} + i \right) \right] = \frac{\pi}{4}$$

$$\text{or } 0 + \arg \left[z - \left(-\frac{1}{2} + i \right) \right] = \frac{\pi}{4}$$

$$\text{or } \arg \left[z - \left(-\frac{1}{2} + i \right) \right] = \frac{\pi}{4}$$

This is a ray that starts from point $-1/2 + i$ and makes an angle $\pi/4$ with the positive real axis as shown in the figure. From the figure, clearly, the system of equations has no solution.

ILLUSTRATION 3.86

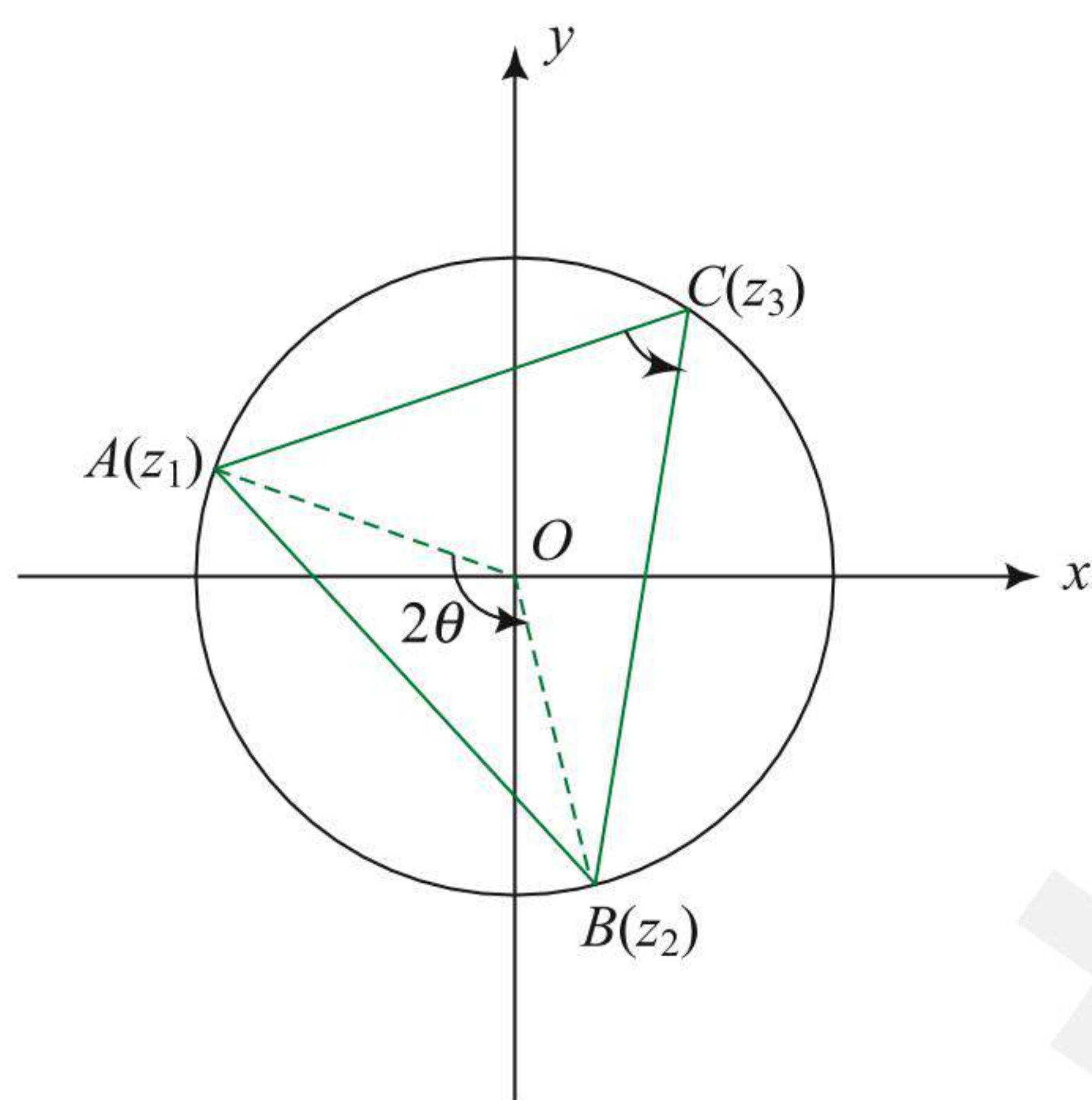
If complex numbers z_1, z_2 and z_3 are such that $|z_1| = |z_2| = |z_3|$,

then prove that $\arg \left(\frac{z_2}{z_1} \right) = \arg \left(\frac{z_2 - z_3}{z_1 - z_3} \right)^2$.

Sol. Given that $|z_1| = |z_2| = |z_3|$

So, z_1, z_2 and z_3 are equidistant from origin.

Thus, origin is circumcenter of the triangle formed by these three complex numbers.



Now, from one of the properties of the chord of the circle, $\angle AOB = 2\angle ACB$

$$\therefore \arg \frac{z_2 - 0}{z_1 - 0} = 2 \arg \frac{z_2 - z_3}{z_1 - z_3}$$

$$\Rightarrow \arg \left(\frac{z_2}{z_1} \right) = \arg \left(\frac{z_2 - z_3}{z_1 - z_3} \right)^2$$

ILLUSTRATION 3.87

If triangle formed by complex numbers z_1, z_2 and z_3 is equilateral then prove that $\frac{z_2 + z_3 - 2z_1}{z_3 - z_2}$ is purely imaginary number.

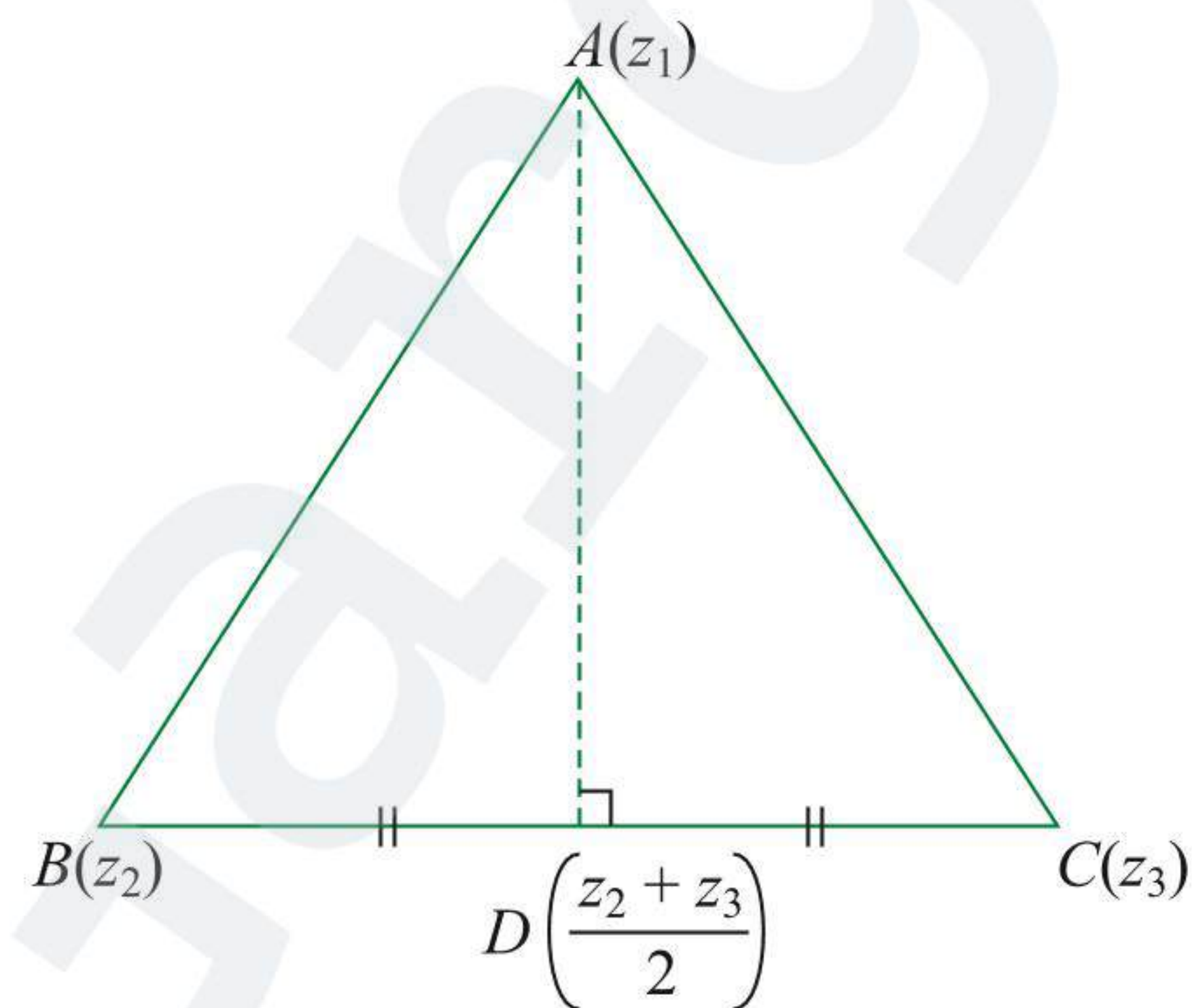
Sol. We have to prove that $\frac{z_2 + z_3 - 2z_1}{z_3 - z_2}$ is purely imaginary number.

$$\text{Then } \arg \left(\frac{z_2 + z_3 - 2z_1}{z_3 - z_2} \right) = \pm \frac{\pi}{2}$$

$$\text{or } \arg 2 \left(\frac{\frac{z_2 + z_3}{2} - z_1}{z_3 - z_2} \right) = \pm \frac{\pi}{2}$$

$$\text{or } \arg 2 + \arg \left(\frac{\frac{z_2 + z_3}{2} - z_1}{z_3 - z_2} \right) = \pm \frac{\pi}{2}$$

$$\text{or } \arg \left(\frac{\frac{z_2 + z_3}{2} - z_1}{z_3 - z_2} \right) = \pm \frac{\pi}{2} \quad [\text{as } \arg(2) = 0]$$



Now, complex number $\frac{z_2 + z_3}{2}$ represents the mid-point of BC , i.e., point D .

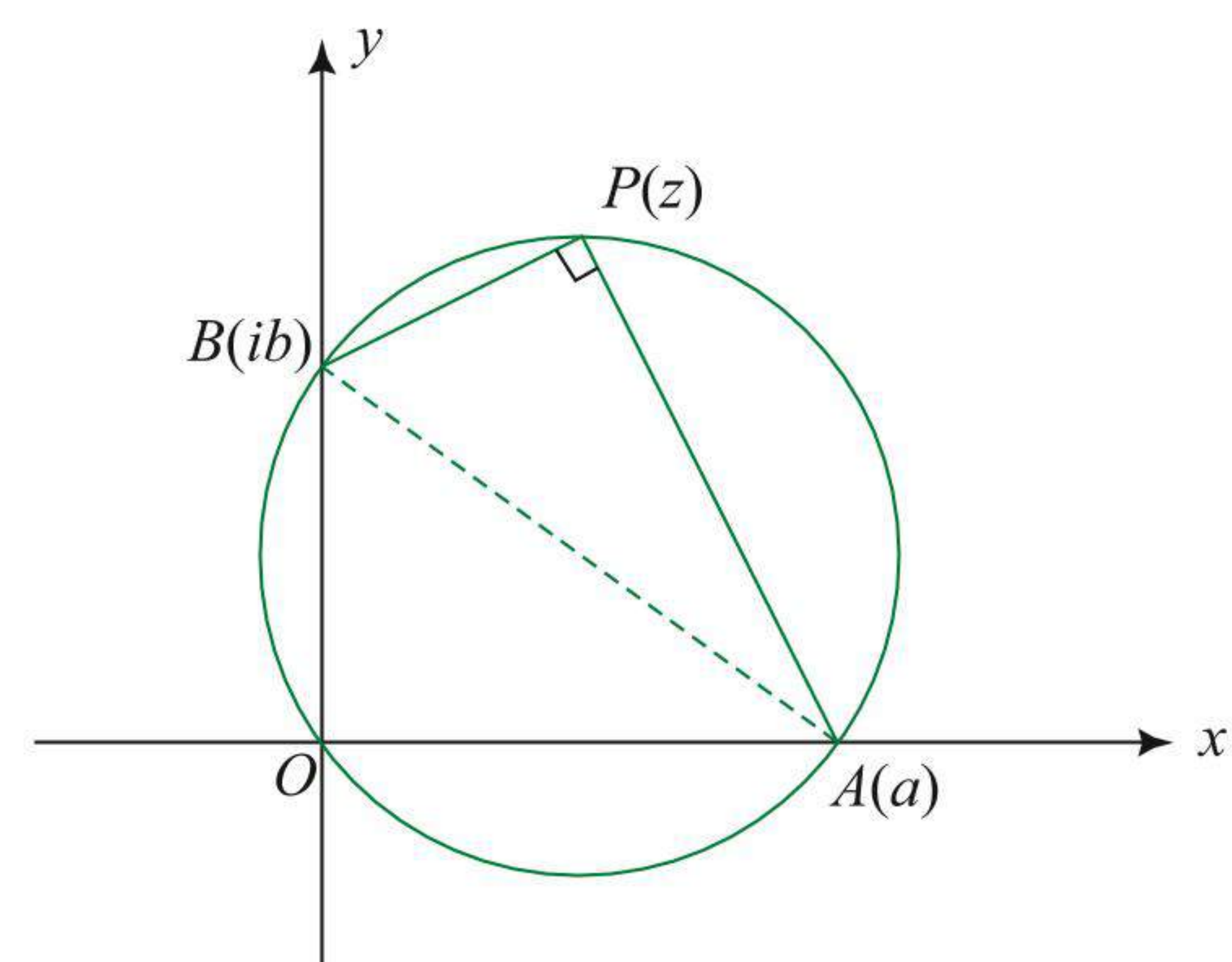
Since triangle ABC is equilateral, $AD \perp BC$.

$$\therefore \arg \left(\frac{\frac{z_2 + z_3}{2} - z_1}{z_3 - z_2} \right) = \pm \frac{\pi}{2}$$

ILLUSTRATION 3.88

Show that the equation of a circle passing through the origin and having intercepts a and b on real and imaginary axis, respectively, on the argand plane is $\operatorname{Re} \left(\frac{z - a}{z - ib} \right) = 0$.

Sol. According to the question, we have circle as shown in the figure.



Clearly, complex numbers ' a ' and ' ib ' are end points of diameter. So, for any complex number z on the circle, we have

$$\arg \left(\frac{z - a}{z - ib} \right) = \pm \frac{\pi}{2}$$

Thus, $\frac{z-a}{z-ib}$ is purely imaginary.

$$\therefore \operatorname{Re}\left(\frac{z-a}{z-ib}\right) = 0$$

ILLUSTRATION 3.89

The triangle formed by $A(z_1)$, $B(z_2)$ and $C(z_3)$ has its circumcentre at origin. If the perpendicular from A to BC intersect the circumference at z_4 then the value of $z_1z_4 + z_2z_3$ is

Sol. $\arg\left(\frac{z_4 - z_1}{z_3 - z_2}\right) = \frac{\pi}{2}$

$$\Rightarrow \frac{z_4 - z_1}{z_3 - z_2} + \frac{\bar{z}_4 - \bar{z}_1}{\bar{z}_3 - \bar{z}_2} = 0$$

$$\Rightarrow \frac{z_4 - z_1}{z_3 - z_2} + \frac{\frac{r^2}{z_3} - \frac{r^2}{z_2}}{\frac{r^2}{z_3} - \frac{r^2}{z_2}} = 0,$$

where r is radius.

$$\Rightarrow \frac{z_4 - z_1}{z_3 - z_2} + \frac{z_2z_3}{z_1z_4} \left(\frac{z_4 - z_1}{z_3 - z_2}\right) = 0$$

$$\Rightarrow z_1z_4 + z_2z_3 = 0$$

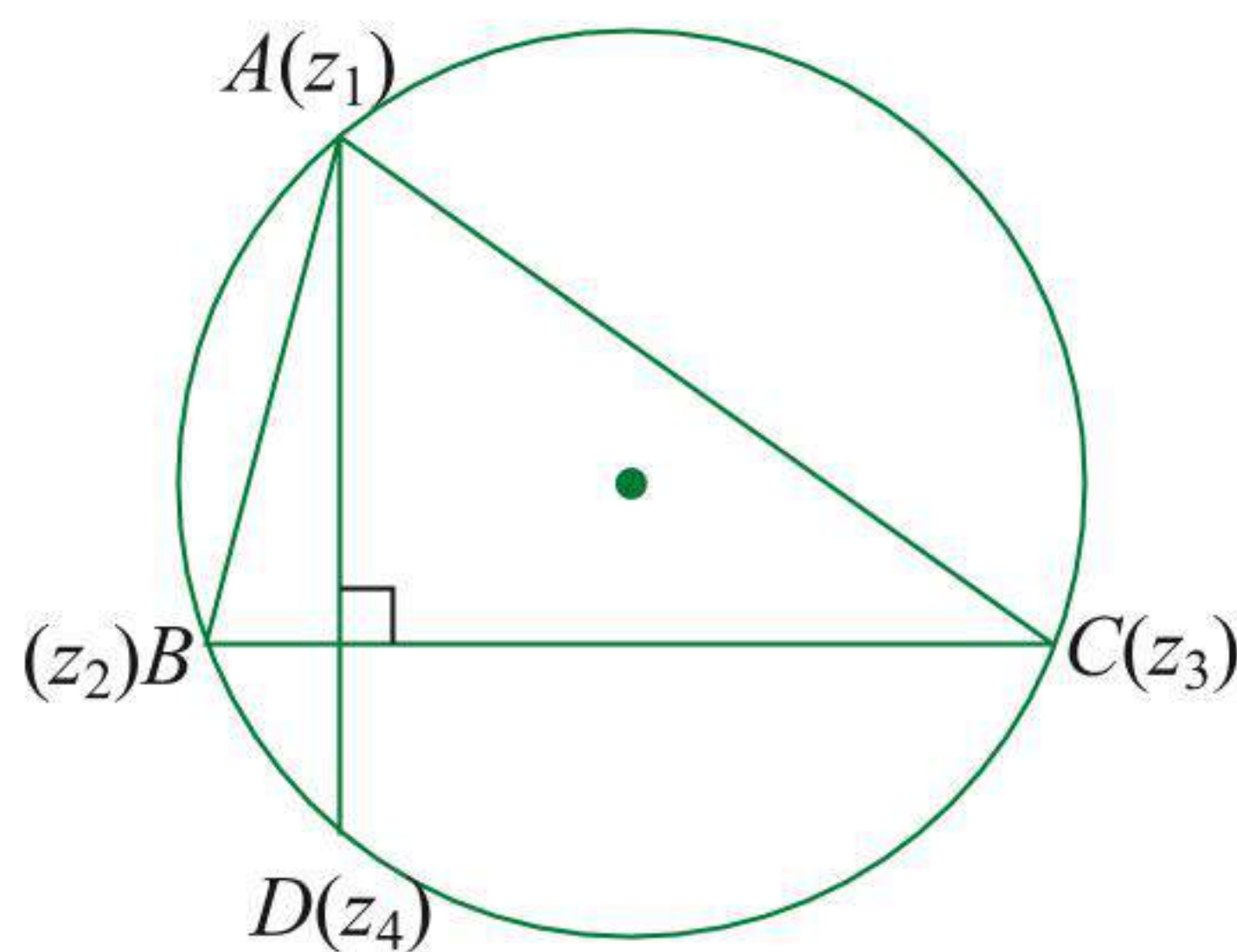
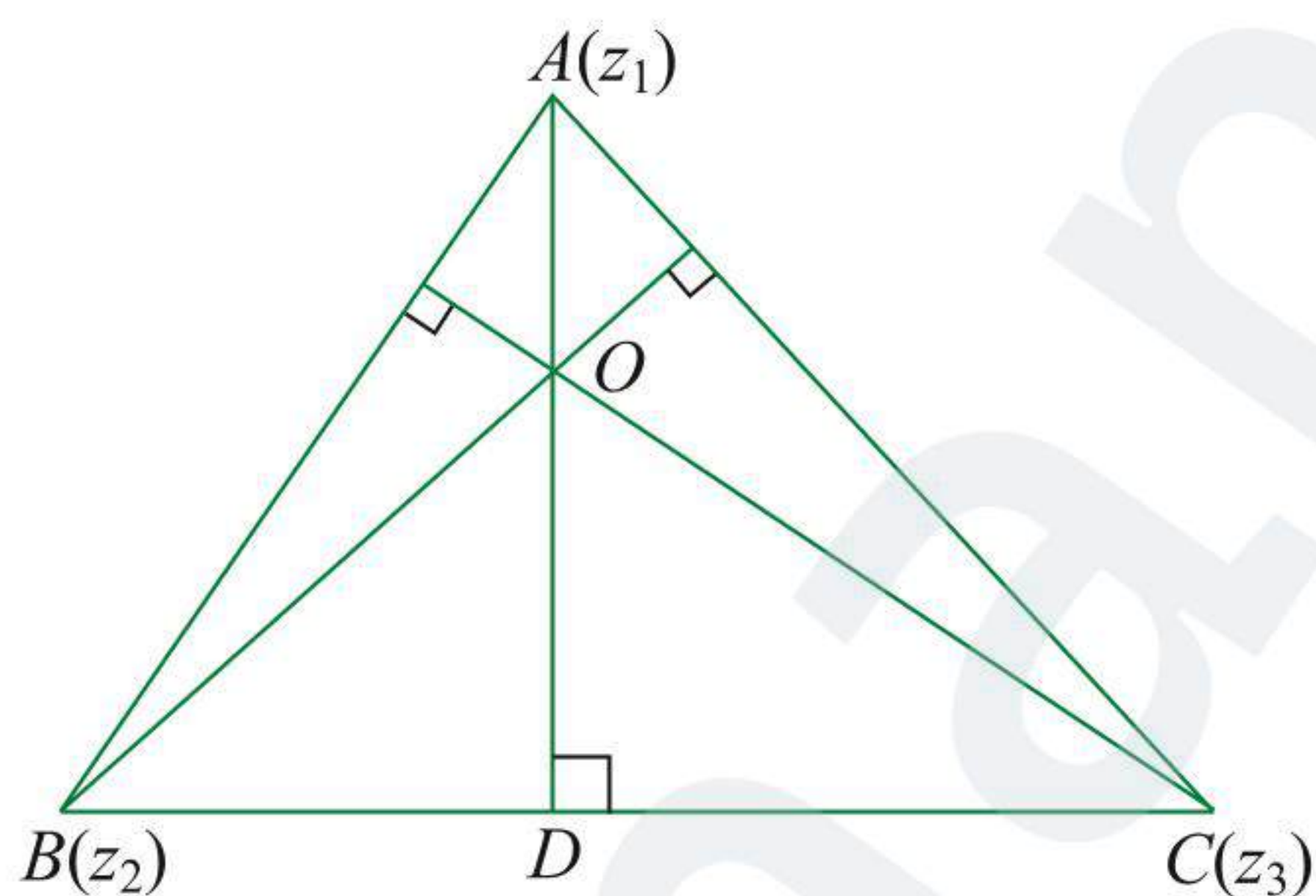


ILLUSTRATION 3.90

Let vertices of an acute-angled triangle are $A(z_1)$, $B(z_2)$, and $C(z_3)$. If the origin O is the orthocenter of the triangle, then prove that $z_1\bar{z}_2 + \bar{z}_1z_2 = z_2\bar{z}_3 + \bar{z}_2z_3 = z_3\bar{z}_1 + \bar{z}_3z_1$.

Sol.



Here, O is orthocenter, then

$$OA \perp BC \quad (\because AD \perp BC)$$

$$\therefore \arg\left(\frac{z_1 - 0}{z_2 - z_3}\right) = \frac{\pi}{2}$$

$$\therefore \frac{z_1 - 0}{z_2 - z_3} \text{ is purely imaginary.}$$

$$\Rightarrow \frac{z_1 - 0}{z_2 - z_3} + \overline{\left(\frac{z_1 - 0}{z_2 - z_3}\right)} = 0$$

$$\Rightarrow \frac{z_1}{z_2 - z_3} + \frac{\bar{z}_1}{\bar{z}_2 - \bar{z}_3} = 0$$

$$\Rightarrow z_1(\bar{z}_2 - \bar{z}_3) + \bar{z}_1(z_2 - z_3) = 0$$

$$\Rightarrow z_1\bar{z}_2 + \bar{z}_1z_2 = z_1\bar{z}_3 + \bar{z}_1z_3 \quad (1)$$

Similarly, $OB \perp AC$

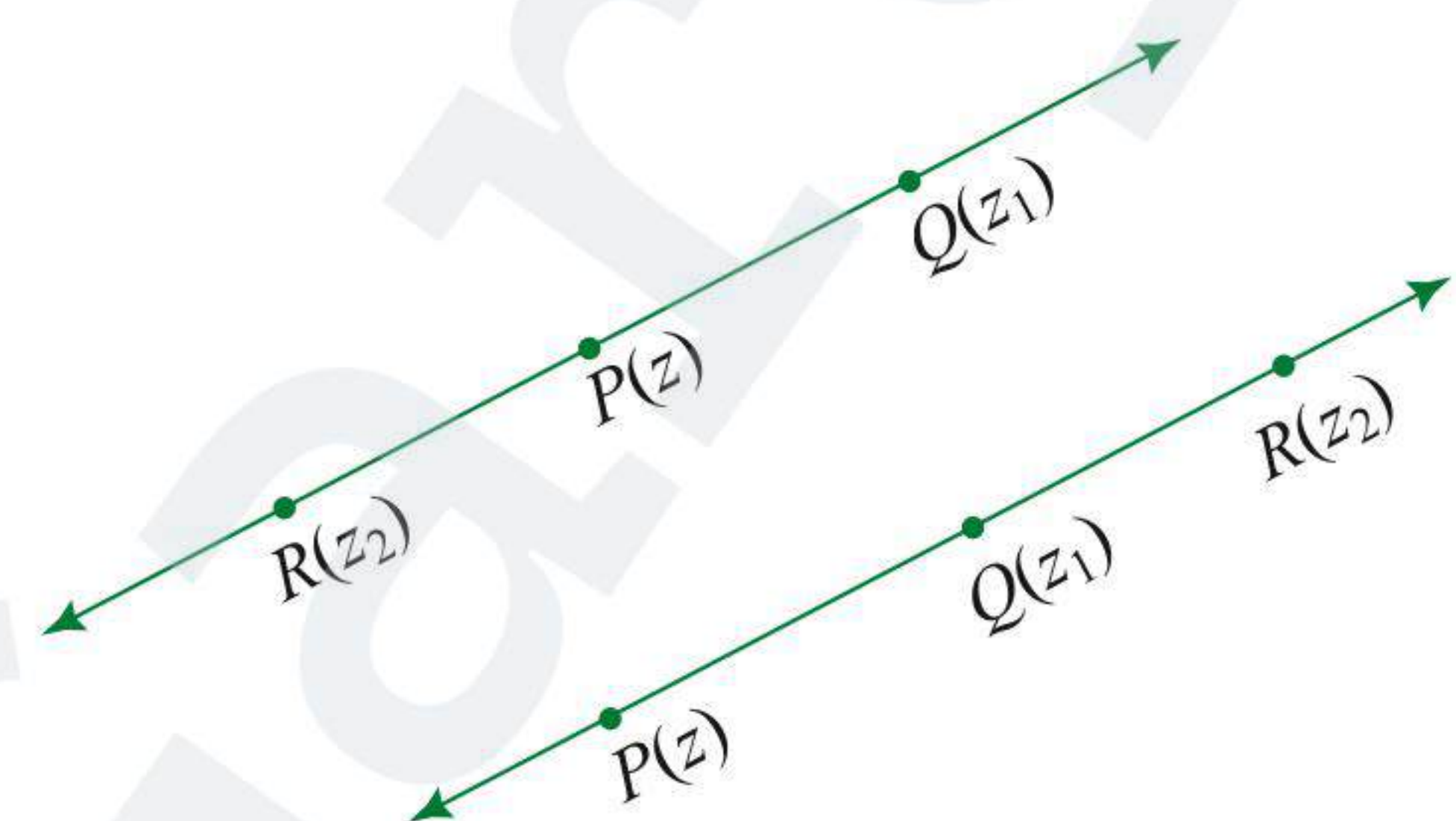
$$\Rightarrow z_1\bar{z}_2 + \bar{z}_1z_2 = z_2\bar{z}_3 + \bar{z}_2z_3 \quad (2)$$

From (1) and (2), $z_1\bar{z}_2 + \bar{z}_1z_2 = z_2\bar{z}_3 + \bar{z}_2z_3 = z_1\bar{z}_3 + \bar{z}_1z_3$

EQUATION OF STRAIGHT LINE

Equation of Line Passing through Complex Numbers z_1 and z_2

Let z be complex number of any point lying on the line passing through the complex numbers z_1 and z_2 .



$$\therefore \arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0, \pi$$

So, $\frac{z - z_1}{z_2 - z_1}$ is purely real

$$\Rightarrow \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \quad (1)$$

$$\Rightarrow (\bar{z}_1 - \bar{z}_2)z - (z_1 - z_2)\bar{z} + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$$

$$\Rightarrow i(\bar{z}_1 - \bar{z}_2)z - i(z_1 - z_2)\bar{z} + i(z_1\bar{z}_2 - z_2\bar{z}_1) = 0$$

$$\Rightarrow \bar{a}z + a\bar{z} + b = 0 \quad (2)$$

where $a = -i(z_1 - z_2)$ and $b = i(z_1\bar{z}_2 - z_2\bar{z}_1)$, $b \in R$.

Equation (1) can also be put in the form

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \quad (3)$$

From above equation, we get the condition for three complex numbers z_1, z_2 and z_3 to be collinear as

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

In equation (2), replacing z by $x + iy$, we get

$$(x + iy)\bar{a} + (x - iy)a + b = 0$$

$$\Rightarrow (a + \bar{a})x + iy(\bar{a} - a) + b = 0$$

$$\text{Slope of line} = -\frac{a + \bar{a}}{i(\bar{a} - a)} = \frac{2\operatorname{Re}(a)}{2i^2\operatorname{Im}(a)} = -\frac{\operatorname{Re}(a)}{\operatorname{Im}(a)}$$

Note:

- Equation of a line parallel to the line $z\bar{a} + \bar{z}a + b = 0$ is $z\bar{a} + \bar{z}a + \lambda = 0$, where $\lambda \in R$.
- Equation of a line perpendicular to the line $z\bar{a} + \bar{z}a + b = 0$ is $z\bar{a} - \bar{z}a + i\lambda = 0$, where $\lambda \in R$.

ILLUSTRATION 3.91

If z_1, z_2, z_3 are three complex numbers such that $5z_1 - 13z_2$

$$+ 8z_3 = 0, \text{ then prove that } \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$$

Sol. $5z_1 - 13z_2 + 8z_3 = 0$

$$\text{or } \frac{5z_1 + 8z_3}{5+8} = z_2$$

This means that z_2 divides segment joining z_1 and z_3 in the ratio 8 : 5. Hence, z_1, z_2, z_3 are collinear.

$$\Rightarrow \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0 \quad (\text{condition of collinear points})$$

ILLUSTRATION 3.92

If $\bar{z} = \bar{z}_0 + A(z - z_0)$, where A is a constant, then prove that locus of z is a straight line.

Sol. $\bar{z} = \bar{z}_0 + A(z - z_0)$

$$\Rightarrow Az - \bar{z} - Az_0 + \bar{z}_0 = 0 \quad (1)$$

$$\Rightarrow \bar{A}\bar{z} - z - \bar{A}\bar{z}_0 + z_0 = 0 \quad (2)$$

Adding (1) and (2), we get

$$(A - 1)z + (\bar{A} - 1)\bar{z} - (Az_0 + \bar{A}\bar{z}_0) + z_0 + \bar{z}_0 = 0$$

This is of the form $\bar{a}z + a\bar{z} + b = 0$, where $a = \bar{A} - 1$ and $b = -(Az_0 + \bar{A}\bar{z}_0) + z_0 + \bar{z}_0 \in \mathbb{R}$. Hence, locus of z is a straight line.

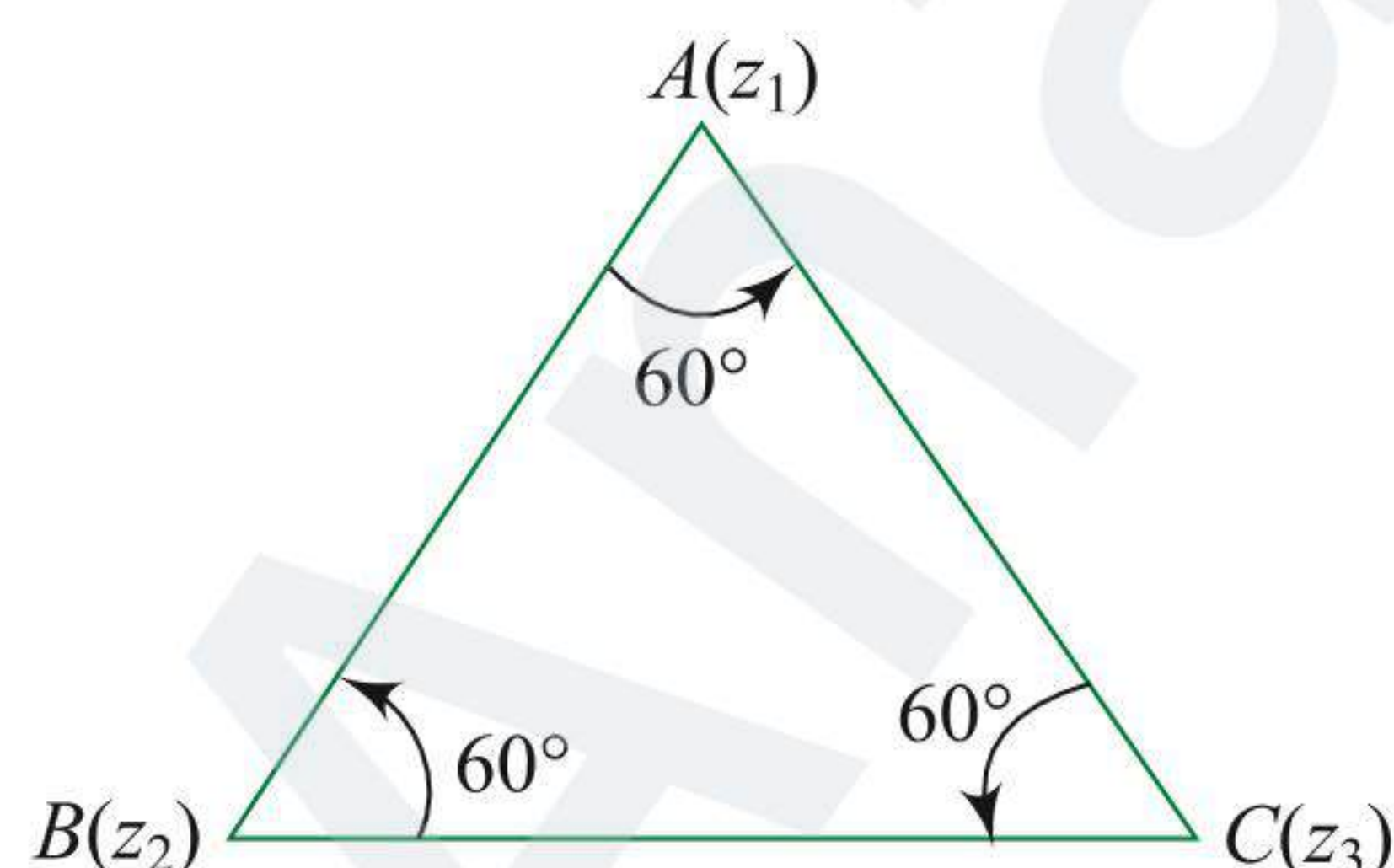
CONDITION FOR THREE POINTS TO REPRESENT EQUILATERAL TRIANGLE

If complex numbers z_1, z_2 and z_3 are the vertices of an equilateral triangle, then

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

$$\text{or } z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Proof:



Since $\triangle ABC$ is equilateral,

$$AB = BC = CA$$

$$\therefore |z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$$

$$\therefore \left| \frac{z_1 - z_2}{z_3 - z_2} \right| = \left| \frac{z_2 - z_3}{z_1 - z_3} \right| \quad (1)$$

Also, $\angle CBA = \angle ACB$

$$\therefore \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right) \quad (2)$$

From (1) and (2), it follows that

$$\begin{aligned} \frac{z_1 - z_2}{z_3 - z_2} &= \frac{z_2 - z_3}{z_1 - z_3} \\ \Rightarrow (z_1 - z_2)(z_1 - z_3) &= -(z_2 - z_3)^2 \\ \Rightarrow z_1^2 - z_1z_2 - z_1z_3 + z_2z_3 &= -(z_2^2 + z_3^2 - 2z_2z_3) \\ \Rightarrow z_1^2 + z_2^2 + z_3^2 &= z_1z_2 + z_2z_3 + z_3z_1 \end{aligned}$$

ILLUSTRATION 3.93

z_1 and z_2 are the roots of $3z^2 + 3z + b = 0$. If $O(0)$, (z_1) , (z_2) form an equilateral triangle, then find the value of b .

Sol. Here, $z_1 + z_2 = -1$, $z_1z_2 = \frac{b}{3}$

Triangle OAB is equilateral. So,

$$0^2 + z_1^2 + z_2^2 = 0 \times z_1 + 0 \times z_2 + z_1z_2$$

$$\text{or } (z_1 + z_2)^2 - 2z_1z_2 = z_1z_2$$

$$\text{or } 1 = 3z_1z_2 = 3 \frac{b}{3}$$

$$\text{or } b = 1$$

ILLUSTRATION 3.94

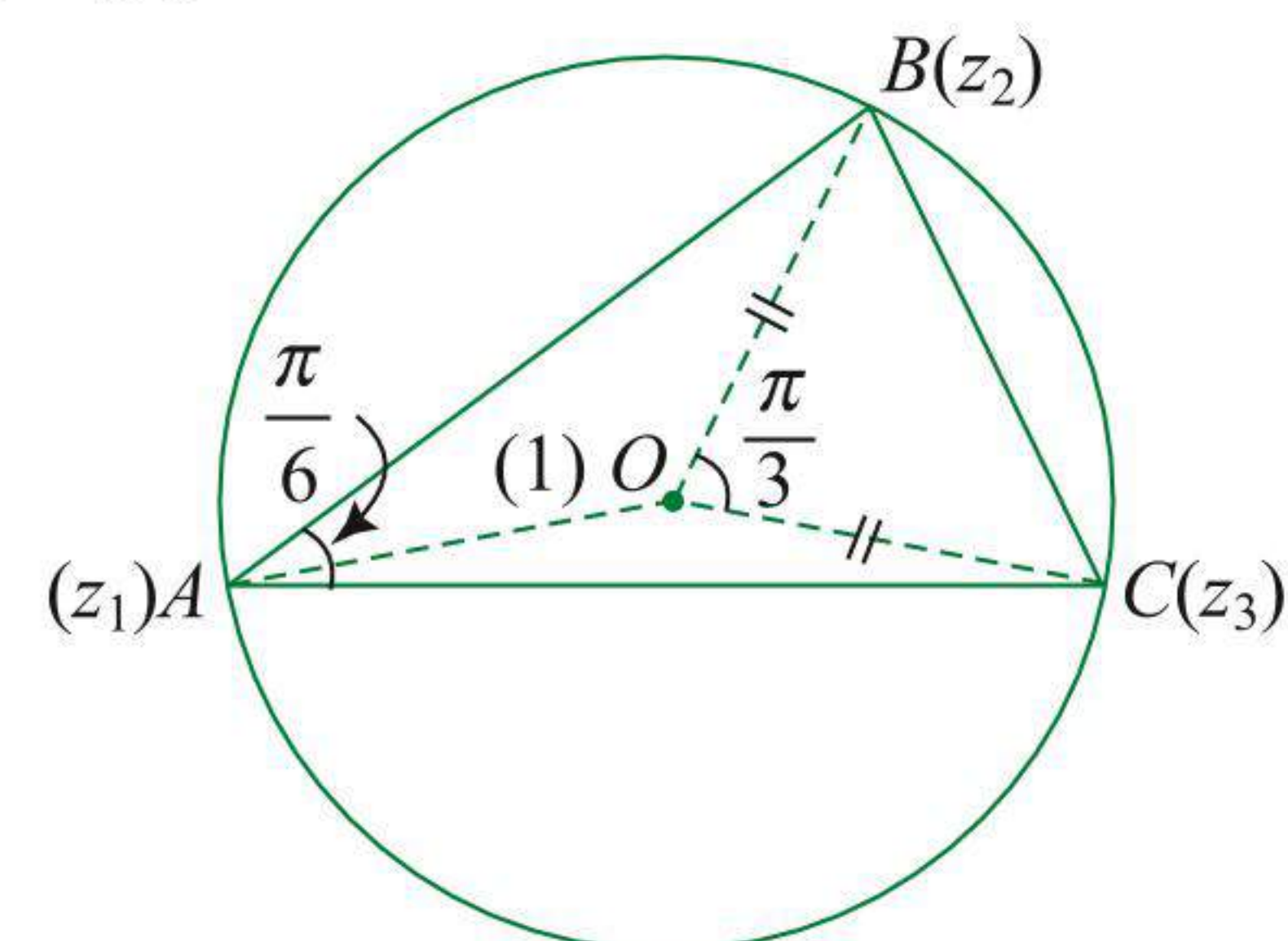
Let z_1, z_2 and z_3 be three complex numbers such that

$$|z_1 - 1| = |z_2 - 1| = |z_3 - 1| \text{ and } \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \frac{\pi}{6},$$

then prove that $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.

Sol. Clearly, z_1, z_2 and z_3 lie on a circle having center at $(1, 0)$.

Also, $\angle BAC = 30^\circ$



So, $\triangle BOC$ is equilateral.

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

ILLUSTRATION 3.95

Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. If z_0 is the circumcentre of the triangle, then prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

Sol. Let $A(z_1)$, $B(z_2)$ and $C(z_3)$ be the vertices of an equilateral triangle.

$$\therefore z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1 \quad (1)$$

$$\begin{aligned} \text{Now, } (z_1 + z_2 + z_3)^2 &= z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1) \\ &= 3(z_1^2 + z_2^2 + z_3^2) \quad (\text{Using (1)}) \end{aligned}$$

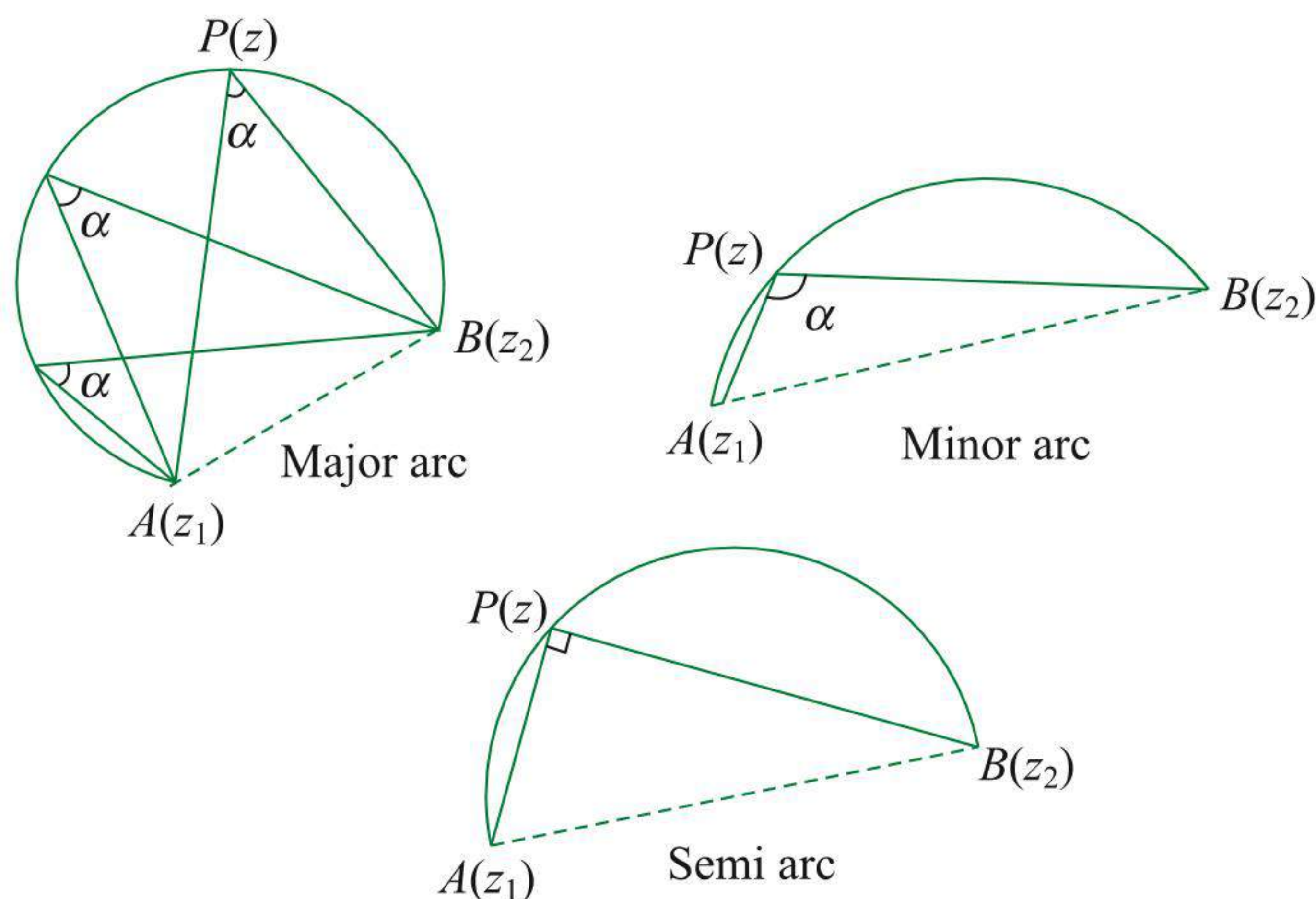
Also, we have $z_0 = \frac{z_1 + z_2 + z_3}{3}$ (as centroid will coincide with circumcentre)

$$(z_1 + z_2 + z_3)^2 = 9z_0^2$$

$$\text{So, } z_1^2 + z_2^2 + z_3^2 = 3z_0^2$$

EQUATION OF ARC

If z_1 and z_2 are fixed complex numbers and z is variable complex number such that $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$ (fixed) then locus of z is an arc as shown in the figure.



Here, $\arg\left(\frac{z - z_1}{z - z_2}\right)$ is angle subtended by line segment

joining z_1 and z_2 at z , which is constant α . This occurs in circle where given chord subtends equal angle at different points on the arc of the circle.

If $\alpha < \pi/2$, we have major arc.

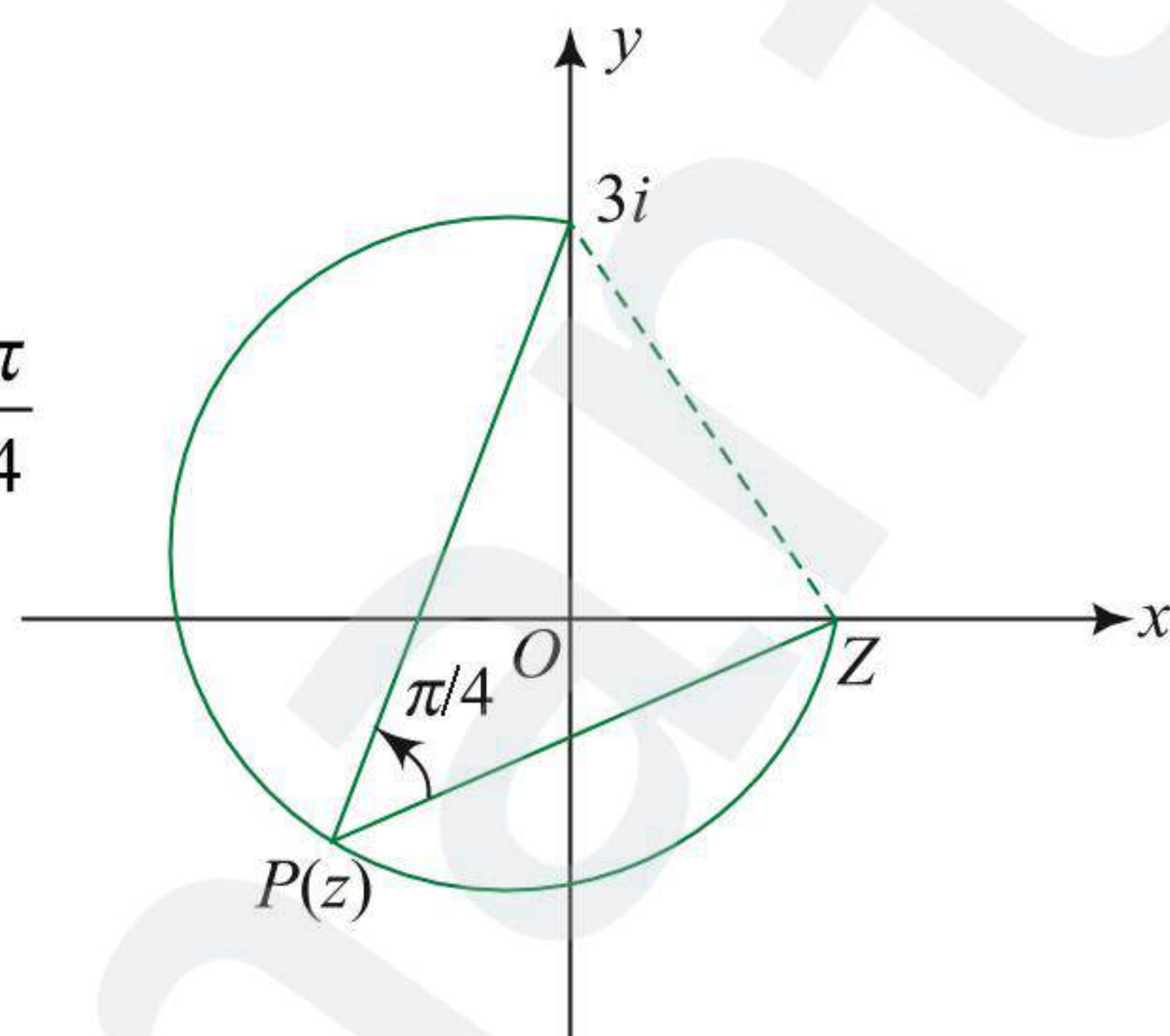
If $\alpha > \pi/2$, we have minor arc.

If $\alpha = \pi/2$, we have semi-arc.

Let us study few figures to understand the orientation of the arc.

$$1. \arg\left(\frac{z - 3i}{z - 2}\right) = \frac{\pi}{4}$$

$$\text{or } \arg\left(\frac{z - 2}{z - 3i}\right) = -\frac{\pi}{4}$$



$$2. \arg\left(\frac{z - 2}{z - 3i}\right) = \frac{\pi}{4} \text{ or } \arg\left(\frac{z - 3i}{z - 2}\right) = -\frac{\pi}{4}$$

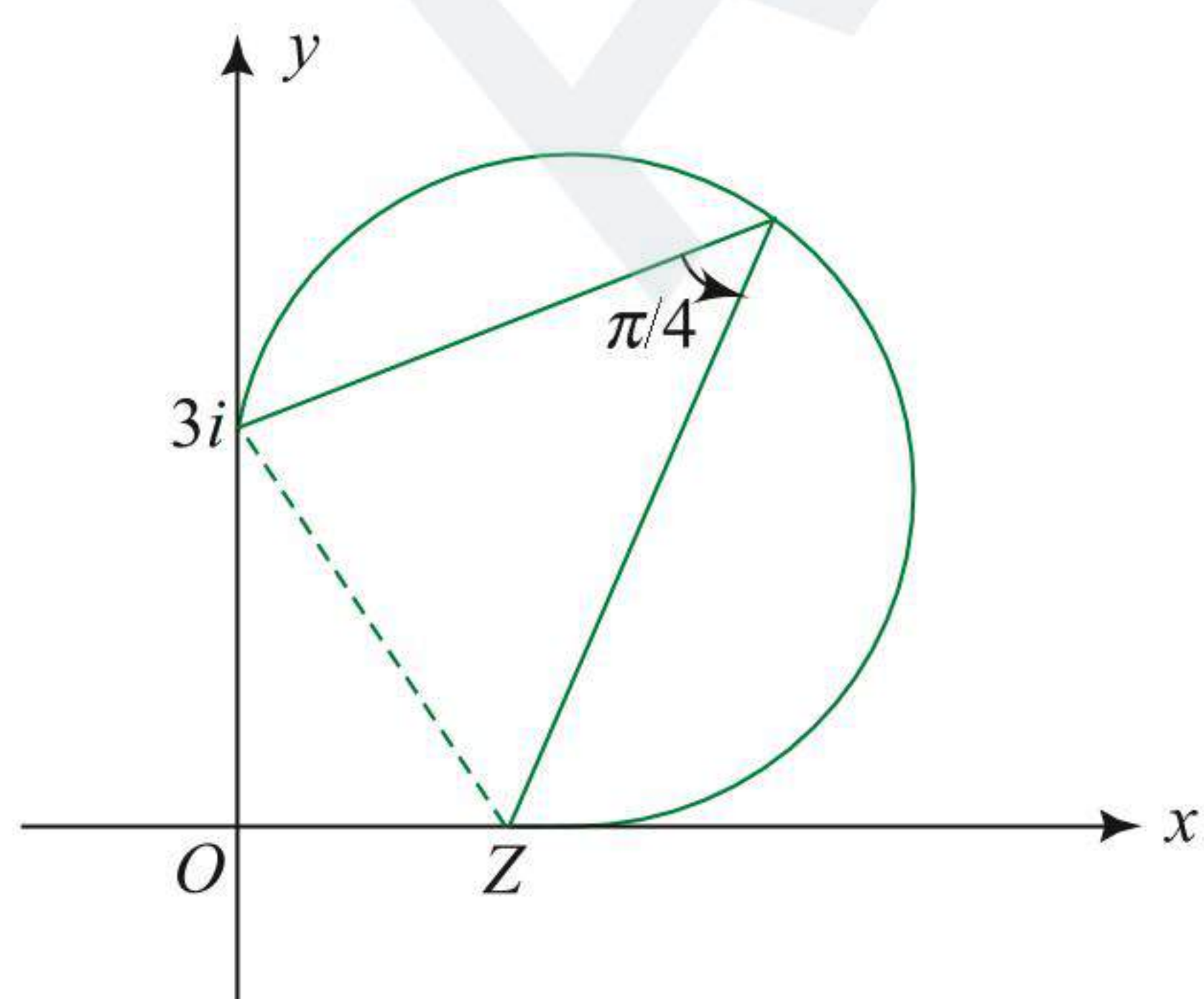


ILLUSTRATION 3.96

In the Argand's plane what is the locus of z ($\neq 1$) such that

$$\arg\left\{\frac{3}{2}\left(\frac{2z^2 - 5z + 3}{3z^2 - z - 2}\right)\right\} = \frac{2\pi}{3}.$$

Sol. $\arg\left\{\frac{3}{2}\left(\frac{2z^2 - 5z + 3}{3z^2 - z - 2}\right)\right\} = \frac{2\pi}{3}$

or $\arg\left\{\frac{3}{2}\frac{(z-1)(2z-3)}{(z-1)(3z+2)}\right\} = \frac{2\pi}{3}$

or $\arg\left\{\frac{3}{2}\left(\frac{2z-3}{3z+2}\right)\right\} = \frac{2\pi}{3}$

or $\arg\left(\frac{z - 3/2}{z + 2/3}\right) = \frac{2\pi}{3}$

Thus, locus of z is minor arc whose end point are $\frac{3}{2}$ and $\frac{-3}{2}$ and included angle is $\frac{2\pi}{3}$.

EQUATION OF CIRCLE WHOSE DIAMETRIC END POINTS ARE GIVEN

Consider the circle having end points of diameter as complex numbers z_1 and z_2 .

Let $P(z)$ be any complex number lying on the circle.

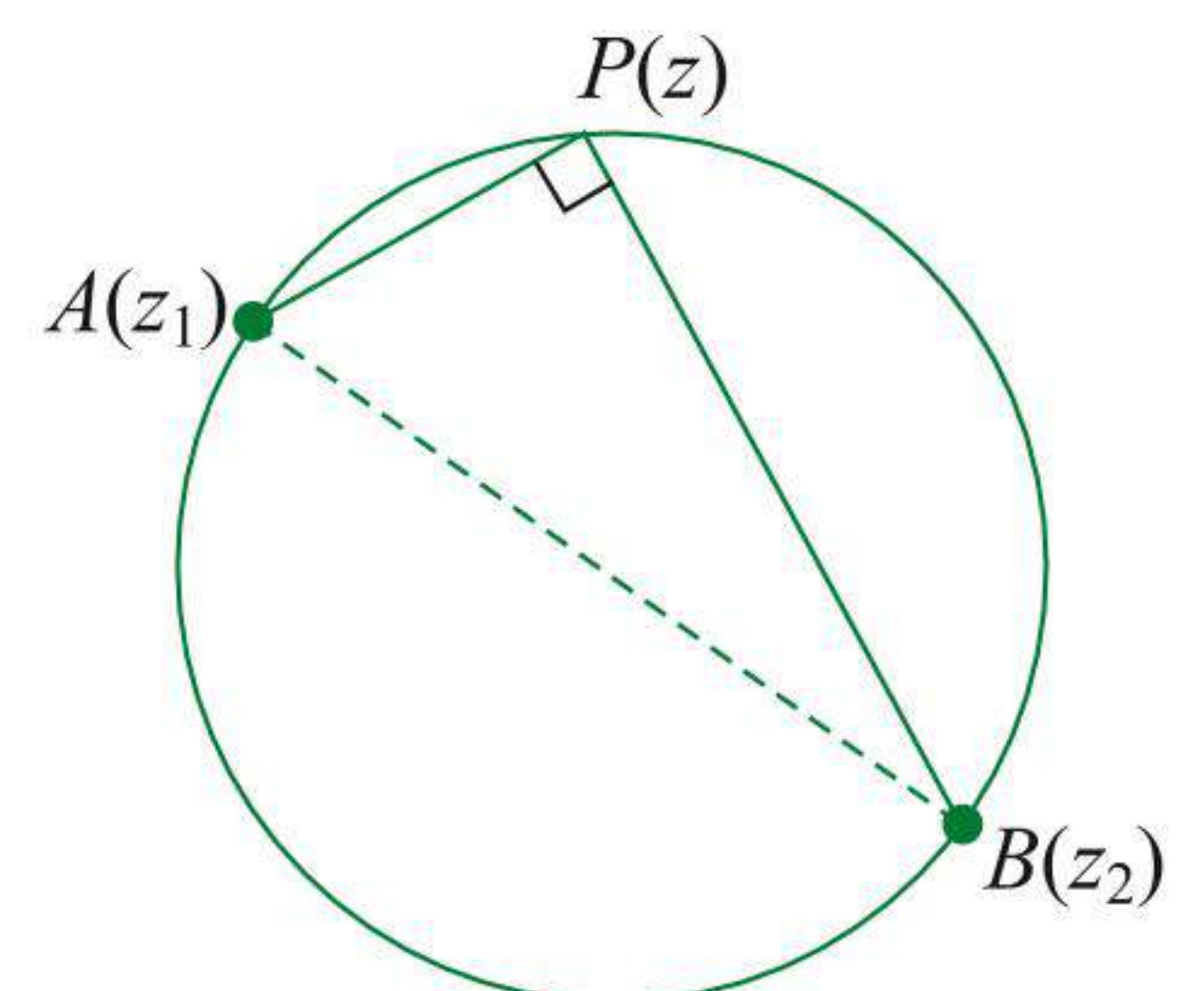
From the figure,

$$\arg\left(\frac{z_1 - z}{z_2 - z}\right) = \pm \pi/2$$

Therefore, $\frac{z - z_1}{z - z_2}$ is purely imaginary.

$$\Rightarrow \frac{z - z_1}{z - z_2} + \frac{\bar{z} - \bar{z}_1}{\bar{z} - \bar{z}_2} = 0$$

$$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$$



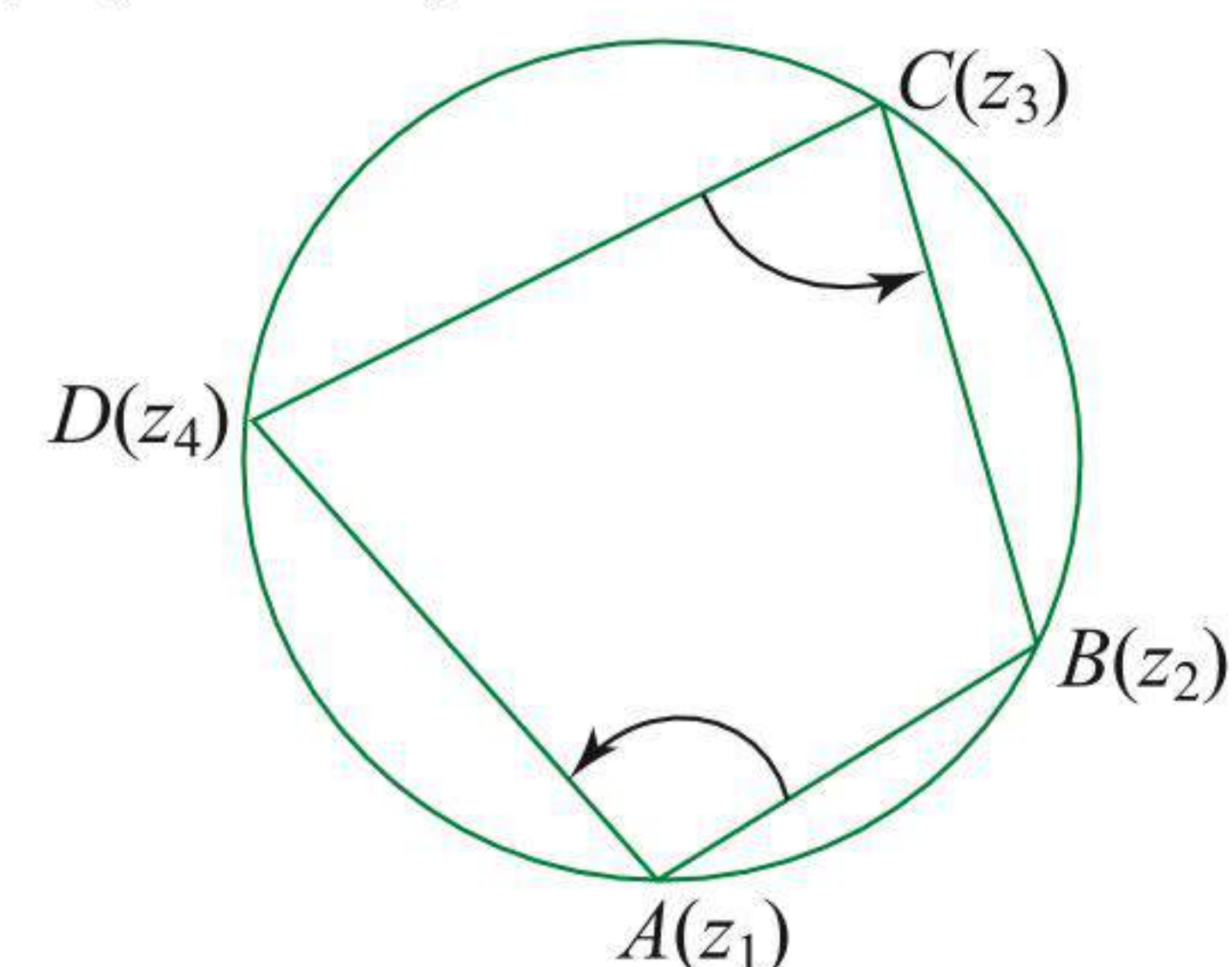
CONDITION FOR FOUR POINTS TO BE CONCYCLIC

Let $ABCD$ be a cyclic quadrilateral such that $A(z_1)$, $B(z_2)$, $C(z_3)$, and $D(z_4)$ lie on a circle. Clearly,

$$\arg\left(\frac{z_4 - z_1}{z_2 - z_1}\right) + \arg\left(\frac{z_2 - z_3}{z_4 - z_3}\right) = \pi$$

$$\Rightarrow \arg\left(\frac{z_4 - z_1}{z_2 - z_1}\right)\left(\frac{z_2 - z_3}{z_4 - z_3}\right) = \pi$$

$$\Rightarrow \left(\frac{z_4 - z_1}{z_2 - z_1}\right)\left(\frac{z_2 - z_3}{z_4 - z_3}\right) \text{ is purely real}$$



Thus, points $A(z_1)$, $B(z_2)$, $C(z_3)$, and $D(z_4)$ (taken in order) would be concyclic if $[(z_4 - z_1)(z_2 - z_3)] / [(z_2 - z_1)(z_4 - z_3)]$ is purely real.

ILLUSTRATION 3.97

If $\left(\frac{3 - z_1}{2 - z_1}\right)\left(\frac{2 - z_2}{3 - z_2}\right) = k$ ($k > 0$), then prove that points $A(z_1)$, $B(z_2)$, $C(3)$, and $D(2)$ (taken in clockwise sense) are concyclic.

Sol. $\left(\frac{3 - z_1}{2 - z_1}\right)\left(\frac{2 - z_2}{3 - z_2}\right) = k$

$\therefore \arg\left(\frac{3 - z_1}{2 - z_1} \cdot \frac{2 - z_2}{3 - z_2}\right) = \arg(k)$

Now $k > 0$. So, $\arg(k) = 0$

$$\therefore \arg\left(\frac{3 - z_1}{2 - z_1}\right) + \arg\left(\frac{2 - z_2}{3 - z_2}\right) = 0$$

$$\text{or } \arg\left(\frac{3 - z_1}{2 - z_1}\right) = \arg\left(\frac{3 - z_2}{2 - z_2}\right)$$

Hence, chord DC subtends same angle at A and B .

Therefore, point A, B, C, D are concyclic.

ILLUSTRATION 3.98

If z_1, z_2, z_3 are complex numbers such that $(2/z_1) = (1/z_2) + (1/z_3)$, then show that the points represented by z_1, z_2, z_3 lie on a circle passing through the origin.

Sol. $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$

or $\frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_3} - \frac{1}{z_1}$

or $\frac{z_2 - z_1}{z_1 z_2} = \frac{z_1 - z_3}{z_3 z_1}$

or $\frac{z_2 - z_1}{z_3 - z_1} = -\frac{z_2}{z_3}$

$\Rightarrow \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(-\frac{z_2}{z_3}\right)$

or $\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \pi + \arg\left(\frac{z_2}{z_3}\right)$

or $\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \pi - \arg\left(\frac{z_3}{z_2}\right)$

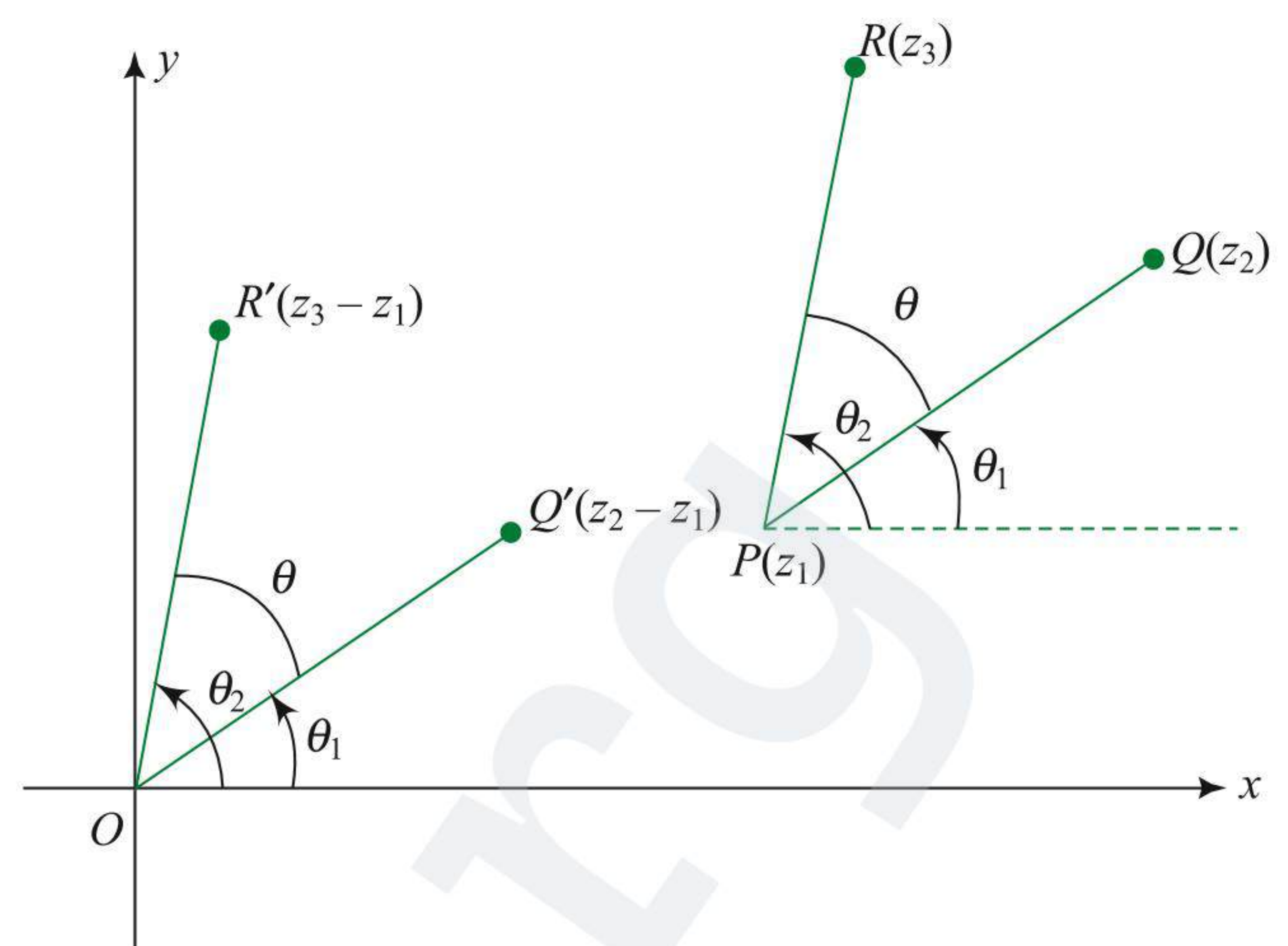
$\Rightarrow \alpha = \pi - \beta$

or $\alpha + \beta = \pi$

Hence, points z_1, z_2, z_3 and 0 are concyclic.

ROTATION FORMULA

Consider the following figure in which PQ is parallel to OQ' and PR is parallel to OR' .



Then, $z_2 - z_1 = |z_2 - z_1|e^{i\theta_1}$ and $z_3 - z_1 = |z_3 - z_1|e^{i\theta_2}$

$$\therefore \frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i(\theta_2 - \theta_1)} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\theta}$$

This formula is called rotation formula.

It gives the complex number of new position $R(z_3)$ attained by $Q(z_2)$ when it is rotated about point $P(z_1)$ and then translated in the direction of new vector PR .

ILLUSTRATION 3.99

$A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of the triangle ABC (in anticlockwise order). If $\angle ABC = \pi/4$ and $AB = \sqrt{2}$ (BC), then prove that $z_2 = z_3 + i(z_1 - z_3)$.

Sol. Rotating about the point B , we get

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{a\sqrt{2}}{a} e^{i\pi/4}$$

$$= \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) = (1 + i)$$

or $z_1 - z_2 = (z_3 - z_2)(1 + i)$

or $z_2(1 - (1 + i)) = z_1 - z_3(1 + i)$

or $z_2 = \frac{z_1}{-i} - \frac{z_3}{-i}(1 + i)$
 $= (iz_1 - iz_3(1 + i)) = z_3 + i(z_1 - z_3)$

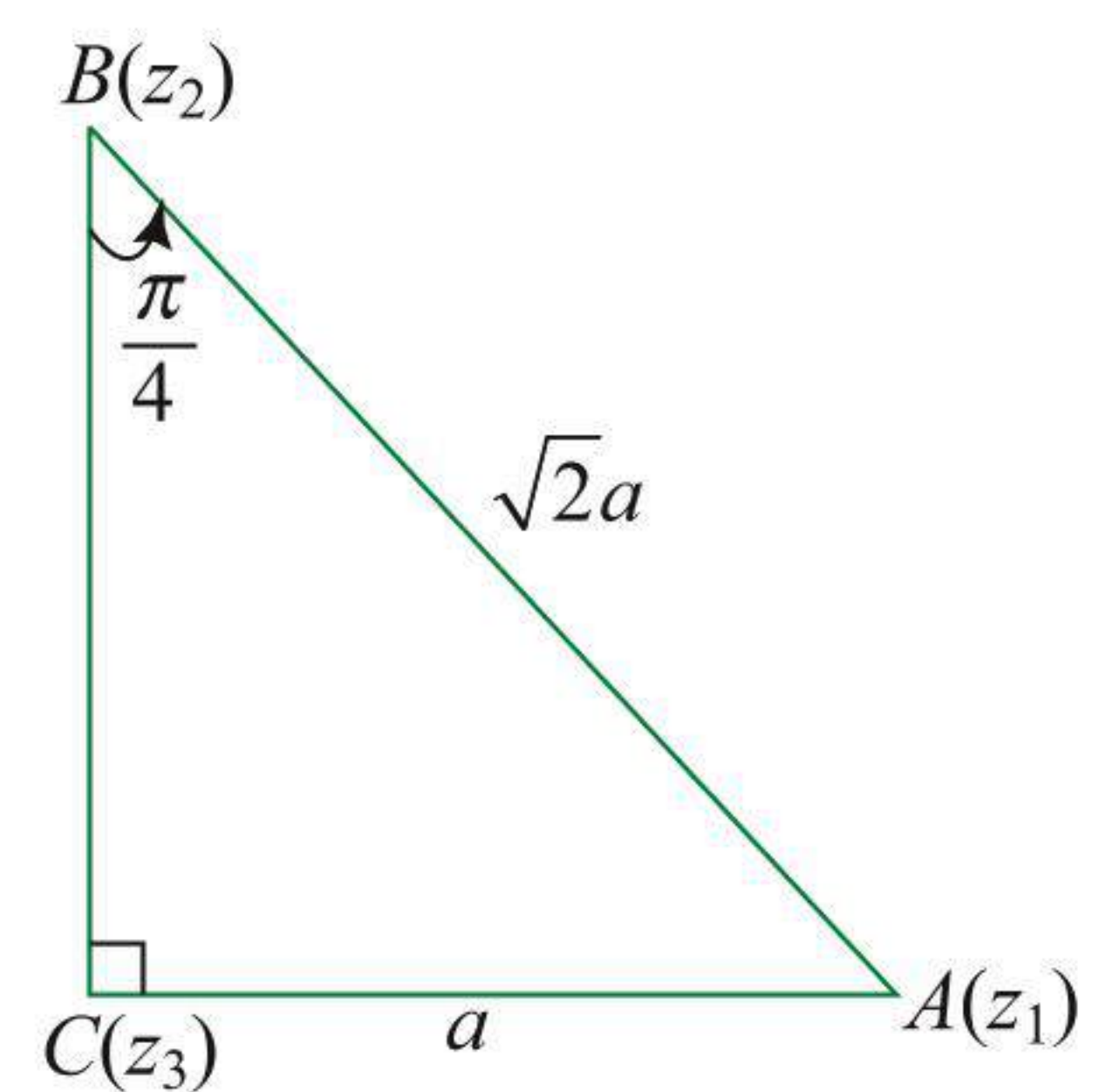
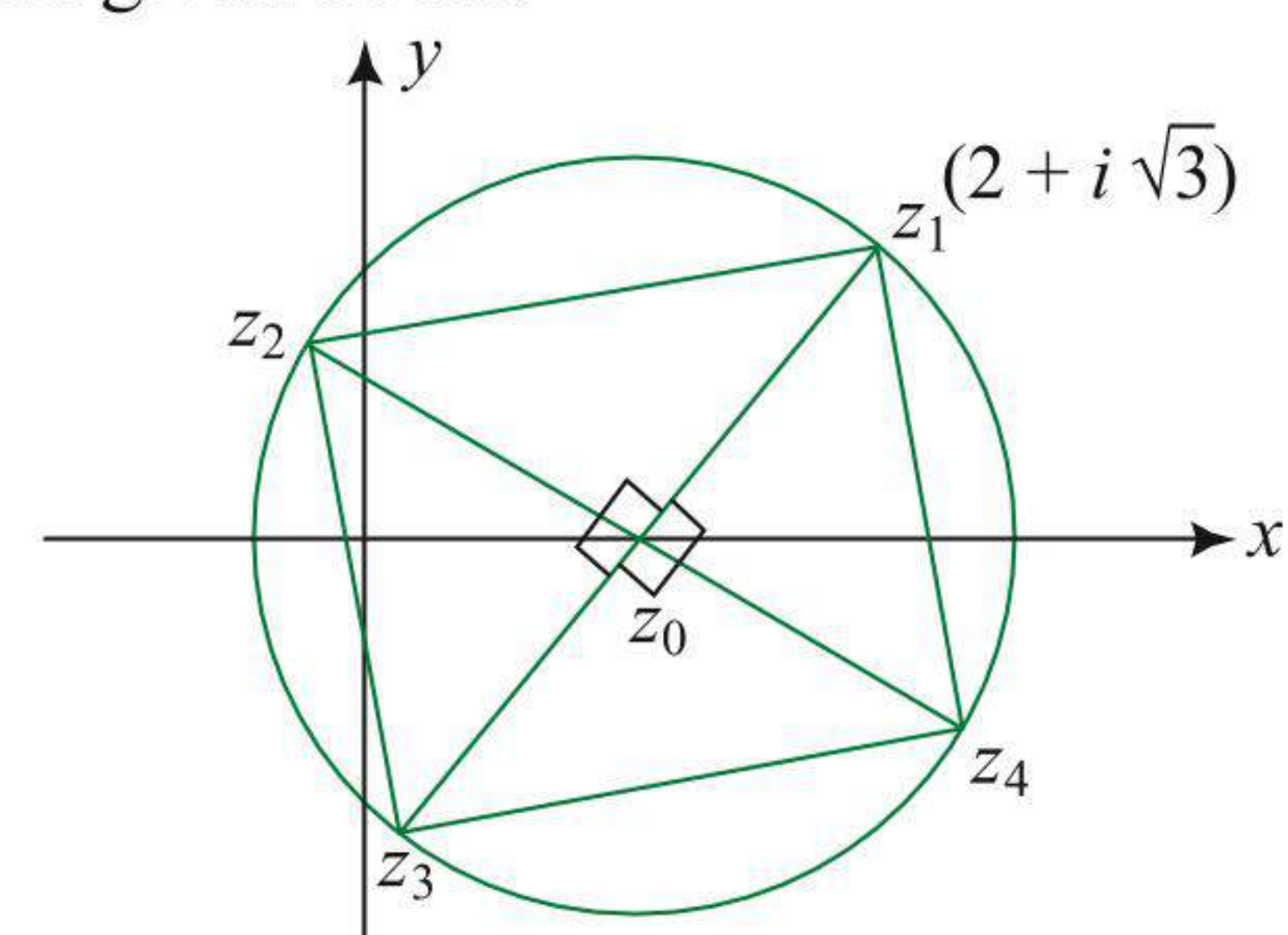


ILLUSTRATION 3.100

If one of the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}i$, find the other vertices of the square.

Sol. The given circle is $|z - 1| = \sqrt{2}$, where $z_0 = 1$ is the center and $\sqrt{2}$ is radius of the circle. z_1 is one of the vertices of the square inscribed in the given circle.



Clearly, z_2 can be obtained by rotating z_1 by an angle of 90° in anticlockwise sense about center z_0 . Thus,

$$\begin{aligned} z_2 - z_0 &= (z_1 - z_0) e^{i\pi/2} \\ \Rightarrow z_2 - 1 &= (2 + i\sqrt{3} - 1)i \\ \Rightarrow z_2 &= i - \sqrt{3} + 1 \\ \Rightarrow z_2 &= (1 - \sqrt{3}) + i \end{aligned}$$

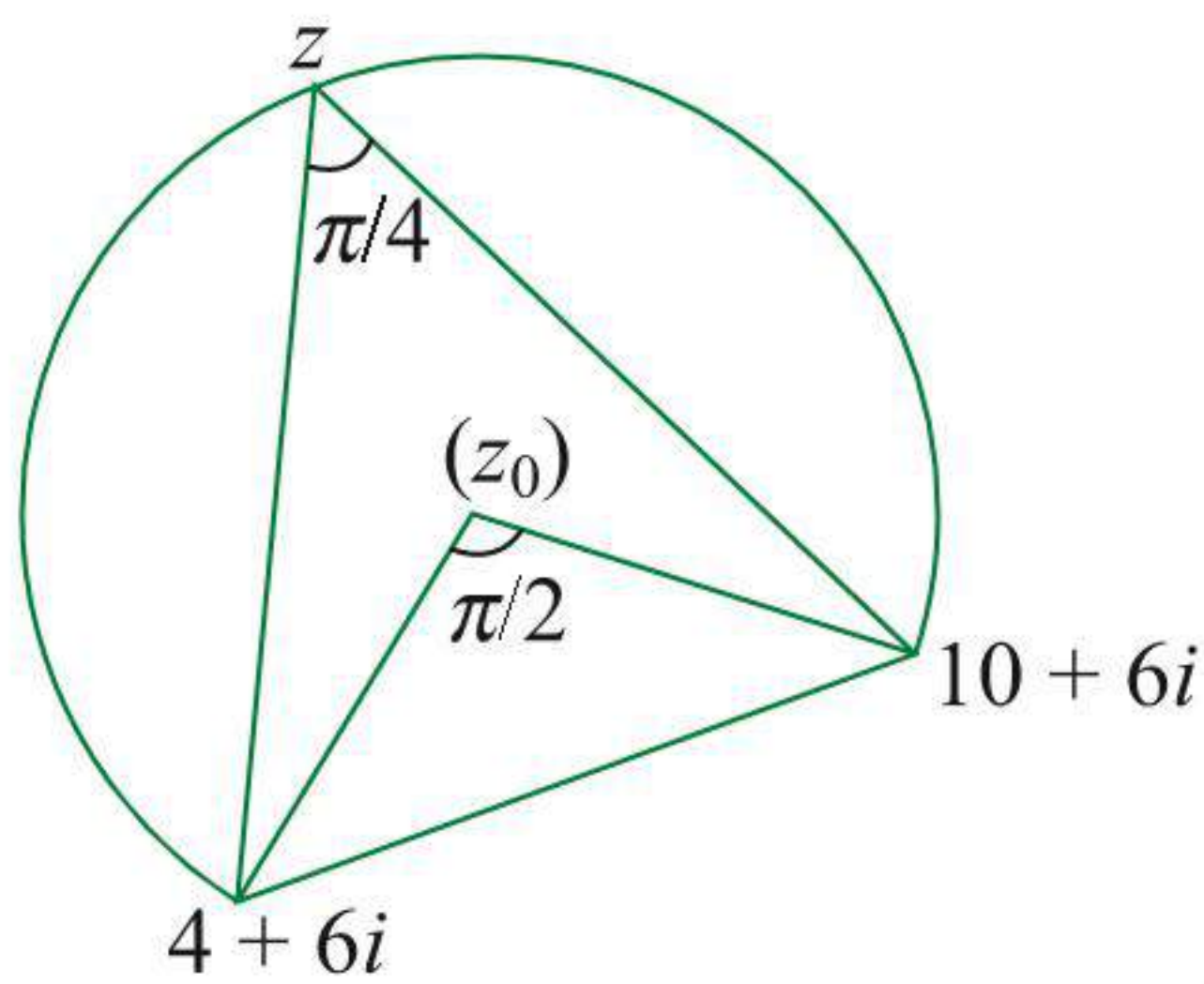
Now z_0 is midpoint of z_1 and z_3 and z_2 and z_4

$$\begin{aligned} \therefore \frac{z_1 + z_3}{2} &= z_0 \Rightarrow \frac{2 + i\sqrt{3} + z_3}{2} = 1 \\ \Rightarrow z_3 &= -i\sqrt{3} \\ \text{and } \frac{z_2 + z_4}{2} &= z_0 \Rightarrow \frac{(1 - \sqrt{3}) + i + z_4}{2} = 1 \\ \Rightarrow z_4 &= (\sqrt{3} + 1) - i \end{aligned}$$

ILLUSTRATION 3.101

Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. Then prove that all the complex numbers satisfying $\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$ also satisfy $|z - 7 - 9i| = 3\sqrt{2}$.

Sol. $\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$



Therefore, z lies on the major arc whose centre is z_0 (say).

Applying rotation at z_0 , we have

$$\begin{aligned} \frac{z_0 - (10 + 6i)}{z_0 - (4 + 6i)} &= \frac{|z_0 - (10 + 6i)|}{|z_0 - (4 + 6i)|} e^{i\frac{\pi}{2}} \\ \Rightarrow \frac{z_0 - (10 + 6i)}{z_0 - (4 + 6i)} &= i \\ \Rightarrow z_0 - 10 - 6i &= iz_0 - 4i + 6 \\ \Rightarrow z_0 &= 7 + 9i \end{aligned}$$

Thus, centre is $7 + 9i$ and z is any point on the arc.

Hence, for any complex number z on the arc, $|z - (7 + 9i)| = |10 + 6i - (7 + 9i)| = 3\sqrt{2}$

ILLUSTRATION 3.102

Complex numbers z_1, z_2, z_3 are the vertices A, B, C , respectively, of an isosceles right-angled triangle with right angle at C . Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Sol. Applying rotation about point A ,

$$\frac{z_1 - z_2}{z_3 - z_2} = \sqrt{2} e^{i\pi/4} \quad (1)$$

Applying rotation about point A ,

$$\frac{z_2 - z_1}{z_3 - z_1} = \sqrt{2} e^{-i\pi/4} \quad (2)$$

Multiplying (1) and (2), we get

$$\begin{aligned} \frac{(z_1 - z_2)(z_2 - z_1)}{(z_3 - z_2)(z_3 - z_1)} &= 2 \\ (z_1 - z_2)^2 &= -2(z_3 - z_2)(z_3 - z_1) \\ &= 2(z_1 - z_3)(z_3 - z_2) \end{aligned}$$

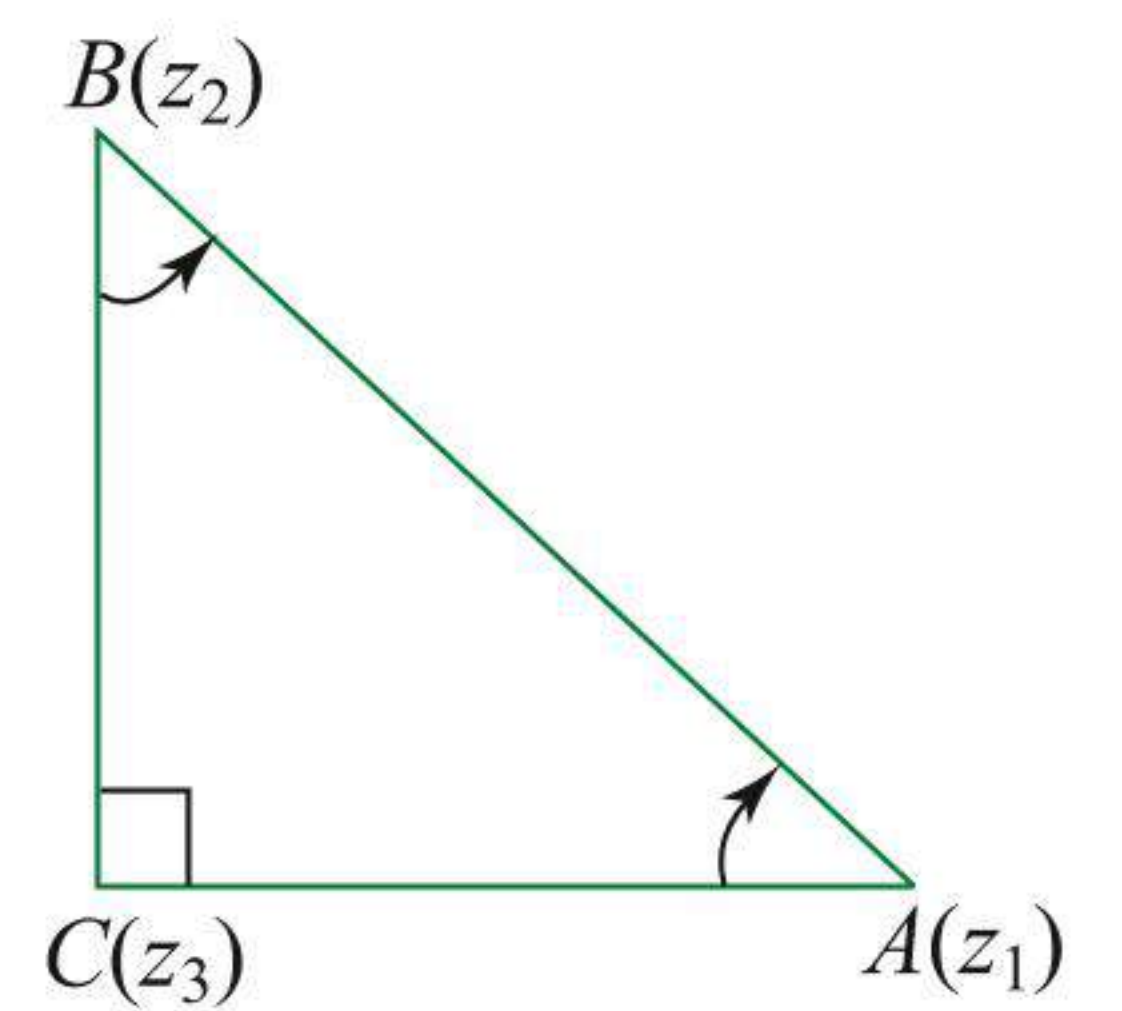


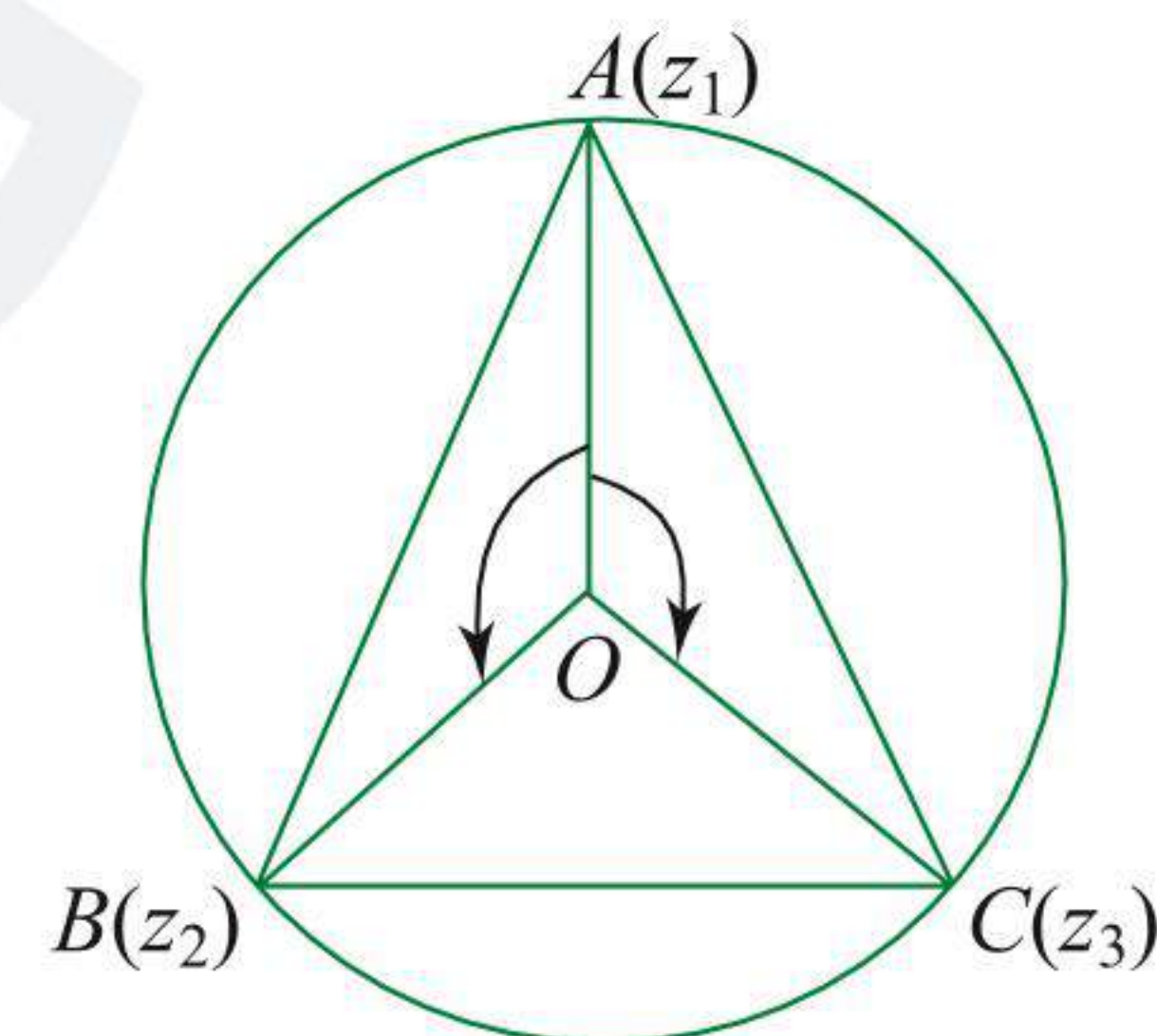
ILLUSTRATION 3.103

Let z_1, z_2 , and z_3 represent the vertices A, B , and C of the triangle ABC , respectively, in the Argand plane, such that $|z_1| = |z_2| = |z_3| = 5$. Prove that $z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C = 0$.

Sol. $|z_1| = |z_2| = |z_3| = 5$

$\Rightarrow |z| = 5$ is the circumcircle of triangle ABC

$\Rightarrow \angle AOB = 2C, \angle BOC = 2A$, and $\angle COA = 2B$



We have,

$$\frac{z_2}{z_1} = \frac{OB}{OA} e^{i2C} = e^{i2C}$$

Similarly, $\frac{z_3}{z_1} = \frac{OC}{OA} e^{-i2B} = e^{-i2B}$

Now, $z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C$

$$\begin{aligned} &= z_1 \left(\sin 2A + \frac{z_2}{z_1} \sin 2B + \frac{z_3}{z_1} \sin 2C \right) \\ &= z_1 (\sin 2A + \sin 2B e^{i2C} + \sin 2C e^{-i2B}) \\ &= z_1 (\sin 2A + \sin 2B \cos 2C + i \sin 2B \sin 2C \\ &\quad + \sin 2C \cos 2B - i \sin 2C \sin 2B) \\ &= z_1 (\sin 2A + \sin(2B + 2C)) \\ &= z_1 (\sin 2A + \sin(2\pi - 2A)) \\ &= 0 \end{aligned}$$

CONCEPT APPLICATION EXERCISE 3.10

- If z_1, z_2, z_3 and z_4 taken in order are vertices of a rhombus, then prove that $\operatorname{Re}\left(\frac{z_3 - z_1}{z_4 - z_2}\right) = 0$.
- Identify the locus of point z if z, i , and iz are collinear.
- If $|z| = 2$ and $\frac{z_1 - z_3}{z_2 - z_3} = \frac{z - 2}{z + 2}$, then prove that z_1, z_2 and z_3 are the vertices of a right angled triangle.

4. Three vertices of triangle are complex numbers α , β and γ . Then prove that the perpendicular from the point α to opposite side is given by the equation $\operatorname{Re}\left(\frac{z-\alpha}{\beta-\gamma}\right) = 0$, where z is complex number of any point on the perpendicular.
5. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2 + z_2^2 - z_1 z_2 = 0$.
6. The center of a regular polygon of n sides is located at the point $z = 0$, and one of its vertices z_1 is known. If z_2 be the vertex adjacent to z_1 , then find z_2 .
7. If one vertex of the triangle having maximum area that can be inscribed in the circle $|z - i| = 5$ is $3 - 3i$, then find the other vertices of the triangle.
8. Consider the circle $|z| = r$ in the Argand plane, which is in fact the incircle of triangle ABC . If contact points opposite to the vertices A, B, C are $A_1(z_1), B_1(z_2)$, and $C_1(z_3)$, obtain the complex numbers associated with the vertices A, B, C in terms of z_1, z_2 , and z_3 .
9. P is a point on the Argand plane. On the circle with OP as the diameter, two points Q and R are taken such that $\angle POQ = \angle QOR = \theta$. If O is the origin and P, Q , and R are represented by the complex numbers z_1, z_2 , and z_3 , respectively, show that $z_2^2 \cos 2\theta = z_1 z_3 \cos^2 \theta$.
10. Find the center of the arc represented by $\arg [(z - 3i)/(z - 2i + 4)] = \pi/4$.

ANSWERS

2. Circle 6. $z_2 = z_1 e^{\pm \frac{i2\pi}{n}}$ 7. $i + (3 - 4i)\left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)$
8. $z_a = \frac{2z_2 z_3}{(z_2 + z_3)}, z_b = \frac{2z_1 z_3}{(z_1 + z_3)}, z_c = \frac{2z_2 z_2}{(z_1 + z_2)}$
10. $\frac{1}{2}(9i - 5)$

THE n TH ROOTS OF UNITY

We know that cube roots of unity are roots of the equation $x^3 = 1$.

$$\therefore x = 1, \frac{-1 + \sqrt{3}i}{2}, \frac{-1 - \sqrt{3}i}{2}$$

$$\therefore x = \cos 0 + i \sin 0, \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}, \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$$

Also, fourth roots of unity are roots of the equation $x^4 = 1$.

$$\therefore x^4 - 1 = 0$$

$$\Rightarrow (x^2 - 1)(x^2 + 1) = 0$$

$$\Rightarrow (x - 1)(x + 1)(x - i)(x + i) = 0$$

$$\Rightarrow x = \pm 1, \pm i$$

$$\Rightarrow x = \cos 0 + i \sin 0, \cos \frac{2\pi}{4} + i \sin \frac{2\pi}{4}, \cos \frac{4\pi}{4} + i \sin \frac{4\pi}{4}, \cos \frac{6\pi}{4} + i \sin \frac{6\pi}{4}$$

Following the above trend, we have fifth roots of unity which are roots of the equation $x^5 = 1$ as

$$\cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5}, k = 0, 1, 2, 3, 4$$

In general, n th roots of unity which are the roots of the equation $x^n = 1$ are given by

$$x = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}; k = 0, 1, 2, \dots, (n - 1)$$

$$\text{Let } \alpha = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}.$$

Then the n th roots of unity are α^t ($t = 0, 1, 2, \dots, n - 1$). So, the n th roots of unity are $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$.

n th roots of -1 :

n th roots of -1 are roots of the equation $x^n = -1$ or $x = (-1)^{1/n}$.

$$\therefore x = (\cos (2k + 1)\pi + i \sin (2k + 1)\pi)^{1/n};$$

$$k = 0, 1, 2, 3, \dots, (n - 1)$$

$$\therefore x = \cos \frac{(2k + 1)\pi}{n} + i \sin \frac{(2k + 1)\pi}{n};$$

$$k = 0, 1, 2, \dots, (n - 1)$$

PROPERTIES OF n TH ROOTS OF UNITY

- When n is odd, there is only one real root '1' of equation $x^n - 1 = 0$.

When n is even, there are two real roots ' ± 1 ' of equation $x^n - 1 = 0$.

- Sum of n th roots of unity**

We have $x^n - 1 = 0$.

$$\text{or } x^n + 0 \times x^{n-1} - 1 = 0$$

$$\text{Sum of the roots} = -\frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n} = 0$$

Also, roots are $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$

$$\therefore S = 1 + \alpha + \alpha^2 + \dots + \alpha^{n-1}$$

$$= \frac{1 - \alpha^n}{1 - \alpha} = \text{Sum of G.P.} = \frac{1 - 1}{1 - \alpha} = 0$$

From this, we can conclude that

$$\sum_{k=0}^{n-1} \cos \frac{2k\pi}{n} = 0 \text{ and } \sum_{k=0}^{n-1} \sin \frac{2k\pi}{n} = 0$$

- Sum of k th power of roots**

$$1^k + \alpha^k + \alpha^{2k} + \dots + \alpha^{(n-1)k}$$

$$= \frac{1 - \alpha^{kn}}{1 - \alpha} = \frac{1 - (\alpha^n)^k}{1 - \alpha} = 0 \quad (\text{as } \alpha^n = 1)$$

- Product of the roots**

$$1 \times \alpha \times \alpha^2 \times \dots \times \alpha^{n-1} = \alpha^{\frac{n(n-1)}{2}}$$

$$= \left(\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} \right)^{n \left(\frac{n-1}{2} \right)}$$

$$= \cos (\pi(n - 1)) + i \sin (\pi(n - 1))$$

$$= \begin{cases} -1, & n \text{ is even} \\ 1, & n \text{ is odd} \end{cases}$$

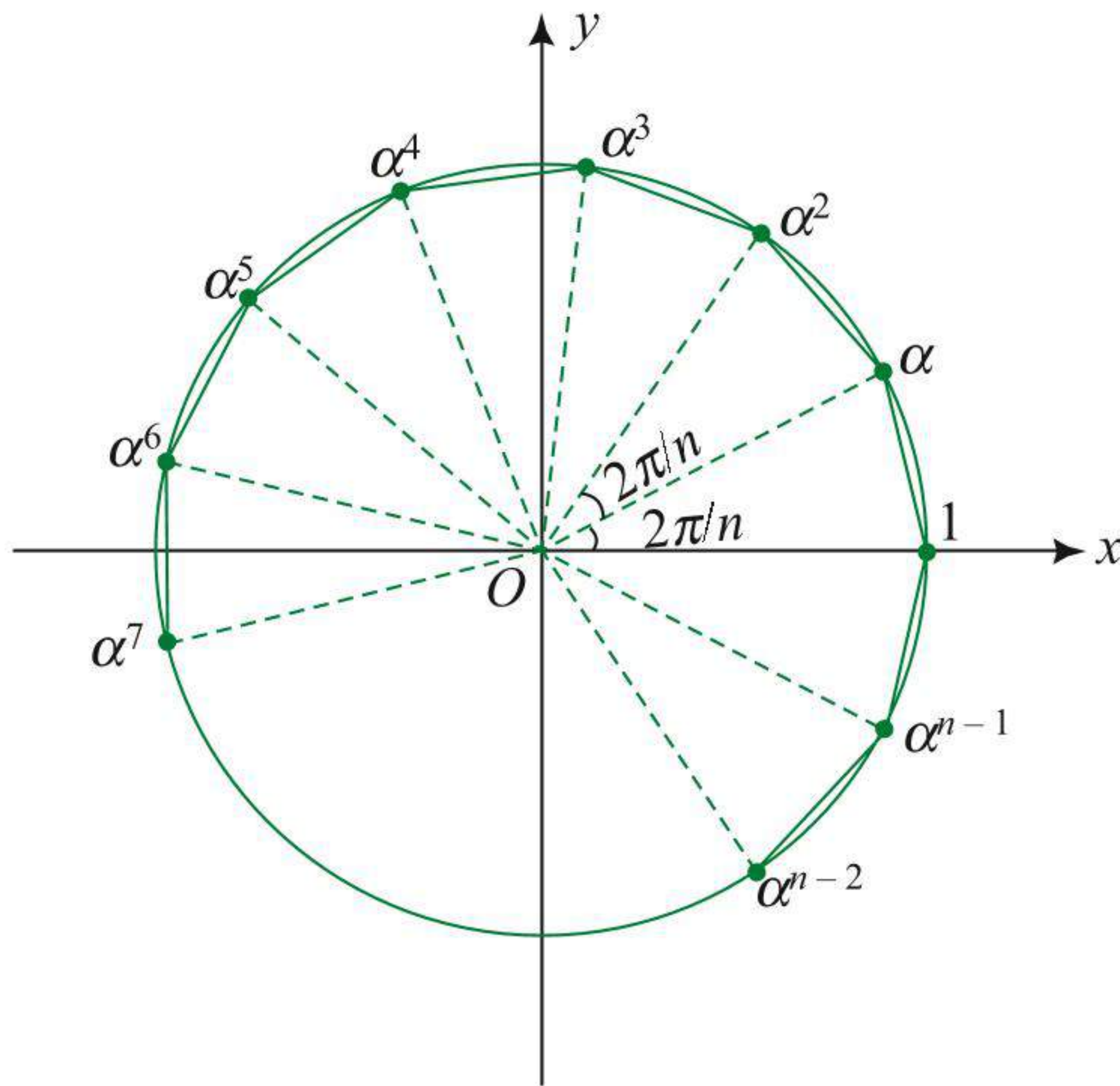
• **Plotting the roots**

The n th roots of unity are given by

$$x = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}; k = 0, 1, 2, 3, \dots, (n-1).$$

Clearly, modulus of this complex number is '1' and argument is $\frac{2k\pi}{n}$ (not necessarily principal argument).

So, these roots lie on unit circle with argument $\frac{2k\pi}{n}$; $k = 0, 1, 2, \dots, (n-1)$.



Thus, n th roots of unity represent vertices of regular polygon of n sides having centre at origin.

• **Common roots**

Consider equations

$$x^p = 1 \quad \dots(1)$$

$$\text{and } x^q = 1 \quad \dots(2)$$

Number of common roots is equal to HCF of p and q .

If p and q are prime numbers then there is only common root, $x = 1$.

ILLUSTRATION 3.104

If $a = \cos(2\pi/7) + i \sin(2\pi/7)$, then find the quadratic equation whose roots are $\alpha = a + a^2 + a^4$ and $\beta = a^3 + a^5 + a^6$.

Sol. $a = \cos(2\pi/7) + i \sin(2\pi/7)$, which is one of the 7th roots of unity.

Therefore, seventh roots of unity are $1, a, a^2, a^3, a^4, a^5$ and a^6 .

$$\text{Also, } a^7 = 1 \quad \dots(1)$$

$$\text{Sum of the roots} = 1 + a + a^2 + a^3 + a^4 + a^5 + a^6 = 0$$

$$\therefore S = \alpha + \beta = (a + a^2 + a^4) + (a^3 + a^5 + a^6) = -1$$

Product of roots,

$$\begin{aligned} P &= \alpha\beta = (a + a^2 + a^4)(a^3 + a^5 + a^6) \\ &= a^4 + a^6 + a^7 + a^5 + a^7 + a^8 + a^7 + a^9 + a^{10} \\ &= a^4 + a^6 + 1 + a^5 + 1 + a + 1 + a^2 + a^3 \quad [\text{from Eq. (1)}] \\ &= 3 + (a + a^2 + a^3 + a^4 + a^5 + a^6) \\ &= 3 - 1 = 2 \end{aligned}$$

Therefore, required equation is: $x^2 - Sx + P = 0$ or $x^2 + x + 2 = 0$

ILLUSTRATION 3.105

If ω is an imaginary fifth root of unity, then find the value of $\log_2 |1 + \omega + \omega^2 + \omega^3 - 1/\omega|$.

Sol. Here, $\omega^5 = 1$. Therefore, $\omega^{-1} = \omega^4$

$$\text{Also, } 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$$

$$\begin{aligned} \therefore \log_2 \left| 1 + \omega + \omega^2 + \omega^3 - \frac{1}{\omega} \right| \\ &= \log_2 |1 + \omega + \omega^2 + \omega^3 - \omega^4| \\ &= \log_2 |-2\omega^4| = \log_2 2 = 1 \quad (\because |\omega| = 1) \end{aligned}$$

ILLUSTRATION 3.106

If $1, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_8$ are ninth roots of unity (taken in counter-clockwise sequence in the Argand plane), then find the value of $|(2 - \alpha_1)(2 - \alpha_3)(2 - \alpha_5)(2 - \alpha_7)|$.

Sol. According to the question,

$$x^9 - 1 = (x - 1)(x - \alpha_1)(x - \alpha_2)(x - \alpha_8)$$

$$\therefore 2^9 - 1 = (2 - \alpha_1)(2 - \alpha_2) \dots (2 - \alpha_8)$$

Since α_1 and α_8 are conjugate to each other, $2 - \alpha_1$ and $2 - \alpha_8$ are also conjugates to each other.

$$\therefore |2 - \alpha_1| = |2 - \alpha_8|$$

Similarly, $|2 - \alpha_2| = |2 - \alpha_7|$, $|2 - \alpha_3| = |2 - \alpha_6|$ and $|2 - \alpha_4| = |2 - \alpha_5|$.

$$\therefore |(2 - \alpha_1)(2 - \alpha_3)(2 - \alpha_5)(2 - \alpha_7)|^2 = 2^9 - 1$$

$$\therefore |(2 - \alpha_1)(2 - \alpha_3)(2 - \alpha_5)(2 - \alpha_7)| = \sqrt{2^9 - 1} = \sqrt{511}$$

ILLUSTRATION 3.107

Find the sum of squares of all roots of the equation $x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0$.

Sol. Given equation is $x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0$.

Multiplying by $x + 1$, we get

$$x^9 + 1 = 0$$

$$\Rightarrow x^9 = -1$$

Hence, the roots are 9th roots of -1 excluding -1 .

Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_8$ be the complex roots, then

$$(-1)^2 + \alpha_1^2 + \alpha_2^2 + \dots + \alpha_8^2 = 0$$

$$\Rightarrow \alpha_1^2 + \alpha_2^2 + \dots + \alpha_8^2 = -1$$

ILLUSTRATION 3.108

Find roots of the equation $(z + 1)^5 = (z - 1)^5$.

Sol. For $z \neq 1$, given equation can be written as $\left(\frac{z+1}{z-1}\right)^5 = 1$.

$$\therefore \frac{z+1}{z-1} = e^{\frac{2k\pi i}{5}}; \text{ where } k = 1, 2, 3, 4$$

$$\Rightarrow z = \frac{e^{2k\pi i/5} + 1}{e^{2k\pi i/5} - 1} = \frac{e^{k\pi i/5} + e^{-k\pi i/5}}{e^{k\pi i/5} - e^{-k\pi i/5}}$$

$$= \frac{2 \cos(k\pi/5)}{2i \sin(k\pi/5)}$$

$$= -i \cot\left(\frac{k\pi}{5}\right); k = 1, 2, 3, 4$$

ILLUSTRATION 3.109

If the roots of $(z-1)^n = i(z+1)^n$ are plotted in the Argand plane, then prove that they are collinear.

Sol. We have $(z-1)^n = i(z+1)^n$

$$\therefore |z-1|^n = |i| |z+1|^n$$

$$\text{or } |z-1|^n = |z+1|^n$$

$$\text{or } |z-1| = |z+1|$$

$$\text{or } |z-1|^2 = |z+1|^2$$

$$\Rightarrow (x-1)^2 + y^2 = (x+1)^2 + y^2$$

$$\text{or } 4x = 0 \text{ or } x = 0.$$

Therefore, the roots lie on the y-axis.

ILLUSTRATION 3.110

Let $1, z_1, z_2, z_3, \dots, z_{n-1}$ be the n th roots of unity. Then prove that $(1-z_1)(1-z_2)\dots(1-z_{n-1}) = n$. Also, deduce that

$$\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} \dots \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}.$$

Sol. $z^n - 1 = (z-1)(z-z_1)(z-z_2)\dots(z-z_{n-1})$

$$\Rightarrow \frac{z^n - 1}{z - 1} = (z - z_1)(z - z_2)\dots(z - z_{n-1})$$

$$\Rightarrow 1 + z + z^2 + \dots + z^{n-1} = (z - z_1)(z - z_2)\dots(z - z_{n-1})$$

Putting $z = 1$, we get

$$(1 - z_1)(1 - z_2)\dots(1 - z_{n-1}) = 1 + 1 + \dots + 1 = n$$

$$\Rightarrow |1 - z_1| |1 - z_2| \dots |1 - z_{n-1}| = n$$

$$\Rightarrow \prod_{r=1}^{n-1} |1 - z_r| = n$$

$$\Rightarrow \prod_{r=1}^{n-1} \sqrt{\left(1 - \cos \frac{2r\pi}{n}\right)^2 + \left(\sin \frac{2r\pi}{n}\right)^2} = n$$

$$\left(\because z_r = \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n} \right)$$

$$\Rightarrow \prod_{r=1}^{n-1} \sqrt{\left(2 \sin^2 \frac{r\pi}{n}\right)^2 + \left(2 \sin \frac{r\pi}{n} \cos \frac{r\pi}{n}\right)^2} = n$$

$$\Rightarrow \prod_{r=1}^{n-1} 2 \sin \frac{r\pi}{n} \sqrt{\sin^2 \frac{r\pi}{n} + \cos^2 \frac{r\pi}{n}} = n$$

$$\Rightarrow \prod_{r=1}^{n-1} 2 \sin \frac{r\pi}{n} = n$$

$$\Rightarrow \prod_{r=1}^{n-1} \sin \left(\frac{r\pi}{n} \right) = \frac{n}{2^{n-1}}$$

CONCEPT APPLICATION EXERCISE 3.11

1. If α is complex fifth root of unity and $(1 + \alpha + \alpha^2 + \alpha^3)^{2005} = p + q\alpha + r\alpha^2 + s\alpha^3$ (where p, q, r, s are real), then find the value of $p + q + r + s$.
2. Find the number of roots of the equation $z^{15} = 1$ satisfying $|\arg z| < \pi/2$.
3. If z is a nonreal root of $\sqrt[7]{-1}$, then find the value of $z^{86} + z^{175} + z^{289}$.
4. Given α, β , respectively, the fifth and the fourth non-real roots of unity, respectively, then find the value of $(1 + \alpha)(1 + \beta)(1 + \alpha^2)(1 + \beta^2)(1 + \alpha^4)(1 + \beta^4)$.
5. If the six solutions of $x^6 = -64$ are written in the form $a + bi$, where a and b are real, then find the product of those solutions with $a > 0$.
6. If $z_r, r = 1, 2, 3, \dots, 50$ are the roots of the equation $\sum_{r=0}^{50} z^r = 0$, then find the value of $\sum_{r=1}^{50} 1/(z_r - 1)$.

ANSWERS

1. -1
2. 7
3. -1
4. 0
5. 4
6. -25

Exercises

Single Correct Answer Type

- If $a < 0, b > 0$, then $\sqrt{a}\sqrt{b}$ is equal to
 - $-\sqrt{|a|b}$
 - $\sqrt{|a|b}i$
 - $\sqrt{|a|b}$
 - none of these
- Consider the equation $10z^2 - 3iz - k = 0$, where z is a complex variable and $i^2 = -1$. Which of the following statements is true?
 - For real positive numbers k , both roots are purely imaginary.
 - For all complex numbers k , neither root is real.
 - For all purely imaginary numbers k , both roots are real and irrational.
 - For real negative numbers k , both roots are purely imaginary.
- The number of solutions of the equation $z^2 + \bar{z} = 0$ is
 - 1
 - 2
 - 3
 - 4
- If center of a regular hexagon is at the origin and one of the vertices on the Argand diagram is $1 + 2i$, then its perimeter is
 - $2\sqrt{5}$
 - $6\sqrt{2}$
 - $4\sqrt{5}$
 - $6\sqrt{5}$
- If x and y are complex numbers, then the system of equations $(1+i)x + (1-i)y = 1, 2ix + 2iy = 1+i$ has
 - unique solution
 - no solution
 - infinite number of solutions
 - none of these
- The points $z_1 = 3 + \sqrt{3}i$ and $z_2 = 2\sqrt{3} + 6i$ are given on a complex plane. The complex number lying on the bisector of the angle formed by the vectors z_1 and z_2 is
 - $z = \frac{(3 + 2\sqrt{3})}{2} + \frac{\sqrt{3} + 2}{2}i$
 - $z = 5 + 5i$
 - $z = -1 - i$
 - none of these
- The polynomial $x^6 + 4x^5 + 3x^4 + 2x^3 + x + 1$ is divisible by (where ω is the cube root of unity)
 - $x + \omega$
 - $x + \omega^2$
 - $(x + \omega)(x + \omega^2)$
 - $(x - \omega)(x - \omega^2)$
 where ω is one of the imaginary cube roots of unity.
- Dividing $f(z)$ by $z - i$, we obtain the remainder i and dividing it by $z + i$, we get the remainder $1 + i$, then remainder upon the division of $f(z)$ by $z^2 + 1$ is
 - $\frac{1}{2}(z + 1) + i$
 - $\frac{1}{2}(iz + 1) + i$
 - $\frac{1}{2}(iz - 1) + i$
 - $\frac{1}{2}(z + i) + 1$
- The complex numbers $\sin x + i \sin 2x$ and $\cos x - i \sin 2x$ are conjugate to each other, for
 - $x = n\pi, n \in \mathbb{Z}$
 - $x = 0$
 - $x = (n + 1/2)\pi, n \in \mathbb{Z}$
 - no value of x
- If the equation $z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$, where a_1, a_2, a_3, a_4 are real coefficients different from zero, has a purely imaginary root, then the expression $a_3/(a_1a_2) + (a_1a_4)/(a_2a_3)$ has the value equal to
 - 0
 - 1
 - 2
 - 2
- If $z_1, z_2 \in \mathbb{C}, z_1^2 + z_2^2 \in \mathbb{R}, z_1(z_1^2 - 3z_2^2) = 2$ and $z_2(3z_1^2 - z_2^2) = 11$, then the value of $z_1^2 + z_2^2$ is
 - 10
 - 12
 - 5
 - 8
- If $a^2 + b^2 = 1$, then $(1 + b + ia)/(1 + b - ia) =$
 - 1
 - 2
 - $b + ia$
 - $a + ib$
- If $z(1 + a) = b + ic$ and $a^2 + b^2 + c^2 = 1$, then $[(1 + iz)/(1 - iz)] =$
 - $\frac{a + ib}{1 + c}$
 - $\frac{b - ic}{1 + a}$
 - $\frac{a + ic}{1 + b}$
 - none of these
- If a and b are complex and one of the roots of the equation $x^2 + ax + b = 0$ is purely real whereas the other is purely imaginary, then
 - $a^2 - (\bar{a})^2 = 4b$
 - $a^2 - (\bar{a})^2 = 2b$
 - $b^2 - (\bar{b})^2 = 4a$
 - $b^2 - (\bar{b})^2 = 2a$
- If $z = (\lambda + 3) - i\sqrt{5 - \lambda^2}$, then the locus of z is
 - ellipse
 - semicircle
 - parabola
 - straight line
- Let $z = 1 - t + i\sqrt{t^2 + t + 2}$, where t is a real parameter. The locus of z in the Argand plane is
 - a hyperbola
 - an ellipse
 - a straight line
 - none of these
- If z_1 and z_2 are the complex roots of the equation $(x - 3)^3 + 1 = 0$, then $z_1 + z_2$ equals to
 - 1
 - 3
 - 5
 - 7
- Which of the following is equal to $\sqrt[3]{-1}$?
 - $\frac{\sqrt{3} + \sqrt{-1}}{2}$
 - $\frac{-\sqrt{3} + \sqrt{-1}}{\sqrt{-4}}$
 - $\frac{\sqrt{3} - \sqrt{-1}}{\sqrt{-4}}$
 - $-\sqrt{-1}$
- If $x^2 + x + 1 = 0$, then the value of $(x + 1/x)^2 + (x^2 + 1/x^2)^2 + \dots + (x^{27} + 1/x^{27})^2$ is
 - 27
 - 72
 - 45
 - 54
- Sum of common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ is
 - 1
 - 1
 - 0
 - 1

21. If $5x^3 + Mx + N$, $M, N \in R$ is divisible by $x^2 + x + 1$, then the value of $M + N$ is
 (1) 5 (2) 4 (3) -4 (4) -5
22. If $z = x + iy$ and $x^2 + y^2 = 16$, then the range of $||x| - |y||$ is
 (1) $[0, 4]$ (2) $[0, 2]$
 (3) $[2, 4]$ (4) none of these
23. If z is a complex number satisfying the equation $z^6 - 6z^3 + 25 = 0$, then the value of $|z|$ is
 (1) $5^{1/3}$ (2) $25^{1/3}$ (3) $125^{1/3}$ (4) $625^{1/3}$
24. If $8iz^3 + 12z^2 - 18z + 27i = 0$, then
 (1) $|z| = \frac{3}{2}$ (2) $|z| = \frac{2}{3}$
 (3) $|z| = 1$ (4) $|z| = \frac{3}{4}$
25. If z_1 and z_2 are complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $[(z_1 + z_2)/(z_1 - z_2)]$ may be
 (1) purely imaginary (2) real and positive
 (3) real and negative (4) none of these
26. If $|z_1| = |z_2|$ and $\arg(z_1/z_2) = \pi$, then $z_1 + z_2$ is equal to
 (1) 0 (2) purely imaginary
 (3) purely real (4) none of these
27. If for complex numbers z_1 and z_2 , $\arg(z_1) - \arg(z_2) = 0$, then $|z_1 - z_2|$ is equal to
 (1) $|z_1| + |z_2|$ (2) $|z_1| - |z_2|$
 (3) $||z_1| - |z_2||$ (4) 0
28. If $\left| \frac{z_1}{z_2} \right| = 1$ and $\arg(z_1 z_2) = 0$, then
 (1) $z_1 = z_2$ (2) $|z_2|^2 = z_1 z_2$
 (3) $z_1 z_2 = 1$ (4) none of these
29. Suppose A is a complex number and $n \in N$, such that $A^n = (A + 1)^n = 1$, then the least value of n is
 (1) 3 (2) 6 (3) 9 (4) 12
30. The number of complex numbers z such that $|z| = 1$ and $|z/\bar{z} + \bar{z}/z| = 1$ is ($\arg(z) \in [0, 2\pi]$)
 (1) 4 (2) 6
 (3) 8 (4) more than 8
31. Let z, w be complex numbers such that $\bar{z} + i\bar{w} = 0$ and $\arg zw = \pi$. Then $\arg z$ equals
 (1) $\frac{\pi}{4}$ (2) $\frac{\pi}{2}$ (3) $\frac{3\pi}{4}$ (4) $\frac{5\pi}{4}$
32. If $z = (3 + 7i)(a + ib)$ where $a, b \in Z - \{0\}$, is purely imaginary, then the minimum value of $|z|^2$ is
 (1) 74 (2) 45 (3) 58 (4) 65
33. If $(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta) \cdots (\cos n\theta + i \sin n\theta) = 1$, then the value of θ is, $m \in Z$
 (1) $4m\pi$ (2) $\frac{2m\pi}{n(n+1)}$ (3) $\frac{4m\pi}{n(n+1)}$ (4) $\frac{m\pi}{n(n+1)}$
34. Given $z = (1 + i\sqrt{3})^{100}$, then $[\operatorname{Re}(z)/\operatorname{Im}(z)]$ equals
 (1) 2^{100} (2) 2^{50} (3) $\frac{1}{\sqrt{3}}$ (4) $\sqrt{3}$
35. The expression $\left[\frac{1 + \sin \frac{\pi}{8} + i \cos \frac{\pi}{8}}{1 + \sin \frac{\pi}{8} - i \cos \frac{\pi}{8}} \right]^8 =$
 (1) 1 (2) -1 (3) i (4) $-i$
36. The number of complex numbers z satisfying $|z - 3 - i| = |z - 9 - i|$ and $|z - 3 + 3i| = 3$ are
 (1) one (2) two
 (3) four (4) none of these
37. $P(z)$ be a variable point in the Argand plane such that $|z| = \min\{|z - 1|, |z + 1|\}$, then $z + \bar{z}$ will be equal to
 (1) -1 or 1 (2) 1 but not equal to -1
 (3) -1 but not equal to 1 (4) none of these
38. If $|z^2 - 1| = |z|^2 + 1$, then z lies on
 (1) a circle (2) a parabola
 (3) an ellipse (4) none of these
39. If $z = x + iy$ ($x, y \in R, x \neq -1/2$), the number of values of z satisfying $|z|^n = z^2 |z|^{n-2} + z |z|^{n-2} + 1$ ($n \in N, n > 1$) is
 (1) 0 (2) 1 (3) 2 (4) 3
40. Number of solutions of the equation $z^3 + [3(\bar{z})^2]/|z| = 0$ where z is a complex number is
 (1) 2 (2) 3 (3) 6 (4) 5
41. Number of ordered pairs (a, b) of real numbers such that $(a + ib)^{2008} = a - ib$ holds good, is
 (1) 2008 (2) 2009 (3) 2010 (4) 1
42. The equation $az^3 + bz^2 + \bar{b}z + \bar{a} = 0$ has a root α ; where a, b, z and α belong to the set of complex numbers. The value of $|\alpha|$
 (1) is $1/2$ (2) is 1
 (3) is 2 (4) can't be determined
43. If $k > 0$, $|z| = |w| = k$ and $\alpha = \frac{z - \bar{w}}{k^2 + z\bar{w}}$, then $\operatorname{Re}(\alpha)$ equals
 (1) 0 (2) $k/2$
 (3) k (4) none of these
44. z_1 and z_2 are two distinct points in an Argand plane. If $a|z_1| = b|z_2|$ (where $a, b \in R$), then the point $(az_1/bz_2) + (bz_2/az_1)$ is a point on the
 (1) line segment $[-2, 2]$ of the real axis
 (2) line segment $[-2, 2]$ of the imaginary axis
 (3) unit circle $|z| = 1$
 (4) the line with $\arg z = \tan^{-1} 2$
45. If z is a complex number such that $-\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2}$, then which of the following inequalities is true?
 (1) $|z - \bar{z}| \leq |z| (\arg z - \arg \bar{z})$
 (2) $|z - \bar{z}| \geq |z| (\arg z - \arg \bar{z})$
 (3) $|z - \bar{z}| < (\arg z - \arg \bar{z})$
 (4) None of these

46. If $\cos \alpha + 2 \cos \beta + 3 \cos \gamma = \sin \alpha + 2 \sin \beta + 3 \sin \gamma = 0$, then the value of $\sin 3\alpha + 8 \sin 3\beta + 27 \sin 3\gamma$ is
 (1) $\sin(\alpha + \beta + \gamma)$ (2) $3 \sin(\alpha + \beta + \gamma)$
 (3) $18 \sin(\alpha + \beta + \gamma)$ (4) $\sin(\alpha + 2\beta + 3\gamma)$
47. If α, β are the roots of the equation $u^2 - 2u + 2 = 0$ and if $\cot \theta = x + 1$, then $[(x + \alpha)^n - (x + \beta)^n] / [\alpha - \beta]$ is equal to
 (1) $\frac{\sin n\theta}{\sin^n \theta}$ (2) $\frac{\cos n\theta}{\cos^n \theta}$
 (3) $\frac{\sin n\theta}{\cos^n \theta}$ (4) $\frac{\cos n\theta}{\sin^n \theta}$
48. If $z = (i)^{(i)^{(i)}}$ where $i = \sqrt{-1}$, then $|z|$ is equal to
 (1) 1 (2) $e^{-\pi/2}$
 (3) $e^{-\pi}$ (4) none of these
49. If $z = i \log(2 - \sqrt{3})$, then $\cos z =$
 (1) -1 (2) -1/2 (3) 1 (4) 1/2
50. If $|z| = 1$, then the point representing the complex number $-1 + 3z$ will lie on
 (1) a circle (2) a straight line
 (3) a parabola (4) a hyperbola
51. The locus of point z satisfying $\operatorname{Re} \left(\frac{1}{z} \right) = k$, where k is a nonzero real number, is
 (1) a straight line (2) a circle
 (3) an ellipse (4) a hyperbola
52. If z is complex number, then the locus of z satisfying the condition $|2z - 1| = |z - 1|$ is
 (1) perpendicular bisector of line segment joining 1/2 and 1
 (2) circle
 (3) parabola
 (4) none of the above curves
53. The greatest positive argument of complex number satisfying $|z - 4| = \operatorname{Re}(z)$ is
 (1) $\frac{\pi}{3}$ (2) $\frac{2\pi}{3}$ (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{4}$
54. If t and c are two complex numbers such that $|t| \neq |c|$, $|t| = 1$ and $z = (at + b)/(t - c)$, $z = x + iy$. Locus of z is (where a, b are complex numbers)
 (1) line segment (2) straight line
 (3) circle (4) none of these
55. If $z^2 + z|z| + |z|^2 = 0$, then the locus of z is
 (1) a circle (2) a straight line
 (3) a pair of straight lines (4) none of these
56. Let C_1 and C_2 are concentric circles of radius 1 and $8/3$, respectively, having center at $(3, 0)$ on the Argand plane. If the complex number z satisfies the inequality $\log_{1/3} \left(\frac{|z - 3|^2 + 2}{11|z - 3| - 2} \right) > 1$, then
 (1) z lies outside C_1 but inside C_2
 (2) z lies inside of both C_1 and C_2
 (3) z lies outside both of C_1 and C_2
 (4) none of these
57. If $|z - 2 - i| = |z| \left| \sin \left(\frac{\pi}{4} - \arg z \right) \right|$, then locus of z is
 (1) a pair of straight lines (2) circle
 (3) parabola (4) ellipse
58. If $|z - 1| \leq 2$ and $|\omega z - 1 - \omega^2| = a$ (where ω is a cube root of unity), then complete set of values of a is
 (1) $0 \leq a \leq 2$ (2) $\frac{1}{2} \leq a \leq \frac{\sqrt{3}}{2}$
 (3) $\frac{\sqrt{3}}{2} - \frac{1}{2} \leq a \leq \frac{1}{2} + \frac{\sqrt{3}}{2}$ (4) $0 \leq a \leq 4$
59. If $|z^2 - 3| = 3|z|$, then the maximum value of $|z|$ is
 (1) 1 (2) $\frac{3 + \sqrt{21}}{2}$
 (3) $\frac{\sqrt{21} - 3}{2}$ (4) none of these
60. If $|2z - 1| = |z - 2|$ and z_1, z_2, z_3 are complex numbers such that $|z_1 - \alpha| < \alpha, |z_2 - \beta| < \alpha$, then $\left| \frac{z_1 + z_2}{\alpha + \beta} \right|$
 (1) $< |z|$ (2) $< 2|z|$ (3) $> |z|$ (4) $> 2|z|$
61. If z_1 is a root of the equation $a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 3$, where $|a_i| < 2$ for $i = 0, 1, \dots, n$, then
 (1) $|z_1| > \frac{1}{3}$ (2) $|z_1| < \frac{1}{4}$
 (3) $|z_1| > \frac{1}{4}$ (4) $|z| < \frac{1}{3}$
62. If $|z| < \sqrt{2} - 1$, then $|z^2 + 2z \cos \alpha|$ is
 (1) less than 1 (2) $\sqrt{2} + 1$
 (3) $\sqrt{2} - 1$ (4) none of these
63. Let $|z_r - r| \leq r, \forall r = 1, 2, 3, \dots, n$. Then $\left| \sum_{r=1}^n z_r \right|$ is less than
 (1) n (2) $2n$
 (3) $n(n+1)$ (4) $\frac{n(n+1)}{2}$
64. All the roots of the equation $11z^{10} + 10iz^9 + 10iz - 11 = 0$ lie
 (1) inside $|z| = 1$ (2) on $|z| = 1$
 (3) outside $|z| = 1$ (4) cannot say
65. If the origin and the non-real roots of $2z^2 + 2z + \lambda = 0$, $\lambda \in \mathbb{R}$ are three vertices of an equilateral triangle in the Argand plane, then λ is
 (1) 1 (2) $\frac{2}{3}$ (3) 2 (4) -1

66. The roots of the equation $t^3 + 3at^2 + 3bt + c = 0$ are z_1, z_2, z_3 which represent the vertices of an equilateral triangle. Then
 (1) $a^2 = 3b$ (2) $b^2 = a$ (3) $a^2 = b$ (4) $b^2 = 3a$
67. The roots of the cubic equation $(z + ab)^3 = a^3$, such that $a \neq 0$, represent the vertices of a triangle of sides of length
 (1) $\frac{1}{\sqrt{3}}|ab|$ (2) $\sqrt{3}|a|$
 (3) $\sqrt{3}|b|$ (4) $|a|$
68. If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$, then the area of the triangle whose vertices are z_1, z_2, z_3 is
 (1) $3\sqrt{3}/4$ (2) $\sqrt{3}/4$ (3) 1 (4) 2
69. Let z and ω be two complex numbers such that $|z| \leq 1$, $|\omega| \leq 1$ and $|z + i\omega| = |z - i\bar{\omega}| = 2$. Then z equals
 (1) 1 or i (2) i or $-i$
 (3) 1 or -1 (4) i or -1
70. Let z_1, z_2 and z_3 be three complex numbers satisfying $|z| = 1$ and $4z_3 = 3(z_1 + z_2)$, then $|z_1 - z_2|$ is equal to
 (1) $\frac{2}{3}$ (2) $\frac{\sqrt{5}}{3}$ (3) $\frac{3}{2}$ (4) $\frac{2\sqrt{5}}{3}$
71. z_1, z_2, z_3, z_4 are distinct complex numbers representing the vertices of a quadrilateral $ABCD$ taken in order. If $z_1 - z_4 = z_2 - z_3$ and $\arg[(z_4 - z_1)/(z_2 - z_1)] = \pi/2$, then the quadrilateral is
 (1) rectangle (2) rhombus
 (3) square (4) trapezium
72. If $k + |k + z^2| = |z|^2$ ($k \in \mathbb{R}^+$), then possible argument of z is
 (1) 0 (2) π
 (3) $\pi/2$ (4) none of these
73. If z_1, z_2, z_3 are the vertices of an equilateral triangle ABC such that $|z_1 - i| = |z_2 - i| = |z_3 - i|$, then $|z_1 + z_2 + z_3|$ equals to
 (1) $3\sqrt{3}$ (2) $\sqrt{3}$ (3) 3 (4) $\frac{1}{3\sqrt{3}}$
74. If z is a complex number having least modulus and $|z - 2 + 2i| = 1$, then $z =$
 (1) $(2 - 1/\sqrt{2})(1 - i)$ (2) $(2 - 1/\sqrt{2})(1 + i)$
 (3) $(2 + 1/\sqrt{2})(1 - i)$ (4) $(2 + 1/\sqrt{2})(1 + i)$
75. If z is a complex number lying in the fourth quadrant of Argand plane and $|[kz/(k+1)] + 2i| > \sqrt{2}$ for all real value of k ($k \neq -1$), then range of $\arg(z)$ is
 (1) $\left(-\frac{\pi}{8}, 0\right)$ (2) $\left(-\frac{\pi}{6}, 0\right)$
 (3) $\left(-\frac{\pi}{4}, 0\right)$ (4) none of these
76. If $|z_2 + iz_1| = |z_1| + |z_2|$ and $|z_1| = 3$ and $|z_2| = 4$, then the area of $\triangle ABC$, if affixes of A, B , and C are z_1, z_2 , and $[(z_2 - iz_1)/(1 - i)]$ respectively, is
 (1) $\frac{5}{2}$ (2) 0 (3) $\frac{25}{2}$ (4) $\frac{25}{4}$
77. If a complex number z satisfies $|2z + 10 + 10i| \leq (5\sqrt{3} - 5)$, then the least principal argument of z is
 (1) $-\frac{5\pi}{6}$ (2) $-\frac{11\pi}{12}$
 (3) $-\frac{3\pi}{4}$ (4) $-\frac{2\pi}{3}$
78. If z lies on the circle $|z - 2i| = 2\sqrt{2}$, then the value of $\arg[(z - 2)/(z + 2)]$ is equal to
 (1) $\frac{\pi}{3}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{2}$
79. z_1 and z_2 lie on a circle with center at the origin. The point of intersection z_3 of the tangents at z_1 and z_2 is given by
 (1) $\frac{1}{2}(\bar{z}_1 + \bar{z}_2)$ (2) $\frac{2z_1z_2}{z_1 + z_2}$
 (3) $\frac{1}{2}\left(\frac{1}{z_1} + \frac{1}{z_2}\right)$ (4) $\frac{z_1 + z_2}{\bar{z}_1\bar{z}_2}$
80. If $\arg\left(\frac{z_1 - \frac{z}{|z|}}{\frac{z}{|z|}}\right) = \frac{\pi}{2}$ and $\left|\frac{z}{|z|} - z_1\right| = 3$, then $|z_1|$ equals to
 (1) $\sqrt{26}$ (2) $\sqrt{10}$ (3) $\sqrt{3}$ (4) $2\sqrt{2}$
81. The maximum area of the triangle formed by the complex coordinates z, z_1, z_2 which satisfy the relations $|z - z_1| = |z - z_2|$ and $|z - (z_1 + z_2)/2| \leq r$, where $r > |z_1 - z_2|$, is
 (1) $\frac{1}{2}|z_1 - z_2|^2$ (2) $\frac{1}{2}|z_1 - z_2|r$
 (3) $\frac{1}{2}|z_1 - z_2|^2 r^2$ (4) $\frac{1}{2}|z_1 - z_2|r^2$
82. Consider the region S of complex numbers a such that $|z^2 - az + 1| = 1$, where $|z| = 1$. Then area of S in the Argand plane is
 (1) $\pi + 8$ (2) $\pi + 4$ (3) $2\pi + 4$ (4) $\pi + 6$
83. If the complex number associated with the vertices A, B, C of $\triangle ABC$ are $e^{i\theta}, \omega, \bar{\omega}$, respectively [where $\omega, \bar{\omega}$ are the complex cube roots of unity and $\cos \theta > \operatorname{Re}(\omega)$], then the complex number of the point where angle bisector of A meets the circumcircle of the triangle is
 (1) $e^{i\theta}$ (2) $e^{-i\theta}$
 (3) $\omega \bar{\omega}$ (4) $\omega + \bar{\omega}$
84. If p and q are distinct prime numbers, then the number of distinct imaginary numbers which are p th as well as q th roots of unity are
 (1) $\min(p, q)$ (2) $\max(p, q)$
 (3) 1 (4) zero
85. Given z is a complex number with modulus 1. Then the equation $[(1 + ia)/(1 - ia)]^4 = z$ has
 (1) all roots real and distinct
 (2) two real and two imaginary
 (3) three roots real and one imaginary
 (4) one root real and three imaginary

86. The value of z satisfying the equation $\log z + \log z^2 + \dots + \log z^n = 0$ is
- $\cos \frac{4m\pi}{n(n+1)} + i \sin \frac{4m\pi}{n(n+1)}, m = 0, 1, 2, \dots$
 - $\cos \frac{4m\pi}{n(n+1)} - i \sin \frac{4m\pi}{n(n+1)}, m = 0, 1, 2, \dots$
 - $\sin \frac{4m\pi}{n} + i \cos \frac{4m\pi}{n}, m = 0, 1, 2, \dots$
 - 0
87. If $n \in \mathbb{N} > 1$, then the sum of real part of roots of $z^n = (z + 1)^n$ is equal to
- $\frac{n}{2}$
 - $\frac{(n-1)}{2}$
 - $-\frac{n}{2}$
 - $\frac{(1-n)}{2}$
88. Which of the following represents a point in an Argand plane, equidistant from the roots of the equation $(z + 1)^4 = 16z^4$?
- (0, 0)
 - $\left(-\frac{1}{3}, 0\right)$
 - $\left(\frac{1}{3}, 0\right)$
 - $\left(0, \frac{2}{\sqrt{5}}\right)$
89. Let a be a complex number such that $|a| < 1$ and $z_1, z_2, z_3, \dots$ be the vertices of a polygon such that $z_k = 1 + a + a^2 + \dots + a^{k-1}$ for all $k = 1, 2, 3, \dots$. Then $z_1, z_2, \dots$ lie within the circle
- $\left|z - \frac{1}{1-a}\right| = \frac{1}{|a-1|}$
 - $\left|z + \frac{1}{a+1}\right| = \frac{1}{|a+1|}$
 - $\left|z - \frac{1}{1-a}\right| = |a-1|$
 - $\left|z + \frac{1}{a+1}\right| = |a+1|$
5. Let a, b and c be distinct non-zero real numbers satisfying $a^3 + b^3 + 6abc = 8c^3$. Then which of the following can be correct? (ω is complex cube root of unity)
- a, c, b are in A.P.
 - a, c, b are in H.P.
 - $a + b\omega - 2c\omega^2 = 0$
 - $a + b\omega^2 - 2c\omega = 0$
6. Let z_1 and z_2 be two non-zero complex numbers such that $|z_1 + z_2| = |z_1| = |z_2|$. Then $\frac{z_1}{z_2}$ can be equal to (ω is imaginary cube root of unity)
- $1 + \omega$
 - $1 + \omega^2$
 - ω
 - ω^2
7. If $p = a + b\omega + c\omega^2$, $q = b + c\omega + a\omega^2$, and $r = c + a\omega + b\omega^2$, where $a, b, c \neq 0$ and ω is the complex cube root of unity, then
- If p, q, r lie on the circle $|z| = 2$, the triangle formed by these points is equilateral.
 - $p^2 + q^2 + r^2 = a^2 + b^2 + c^2$
 - $p^2 + q^2 + r^2 = 2(pq + qr + rp)$
 - none of these
8. Let $P(x)$ and $Q(x)$ be two polynomials. Suppose that $f(x) = P(x^3) + xQ(x^3)$ is divisible by $x^2 + x + 1$, then
- $P(x)$ is divisible by $(x - 1)$, but $Q(x)$ is not divisible by $x - 1$
 - $Q(x)$ is divisible by $(x - 1)$, but $P(x)$ is not divisible by $x - 1$
 - Both $P(x)$ and $Q(x)$ are divisible by $x - 1$
 - $f(x)$ is divisible by $x - 1$
9. If α is a complex constant such that $\alpha z^2 + z + \bar{\alpha} = 0$ has a real root, then
- $\alpha + \bar{\alpha} = 1$
 - $\alpha + \bar{\alpha} = 0$
 - $\alpha + \bar{\alpha} = -1$
 - the absolute value of the real root is 1
10. If $z^3 + (3 + 2i)z + (-1 + ia) = 0$ has one real root, then the value of a lies in the interval ($a \in \mathbb{R}$)
- $(-2, 1)$
 - $(-1, 0)$
 - $(0, 1)$
 - $(-2, 3)$
11. Given that the complex numbers which satisfy the equation $|z\bar{z}^3| + |\bar{z}z^3| = 350$ form a rectangle in the Argand plane with the length of its diagonal having an integral number of units, then
- area of rectangle is 48 sq. units
 - if z_1, z_2, z_3, z_4 are vertices of rectangle, then $z_1 + z_2 + z_3 + z_4 = 0$
 - rectangle is symmetrical about the real axis
 - None of these
12. If the points $A(z)$, $B(-z)$, and $C(1 - z)$ are the vertices of an equilateral triangle ABC , then
- sum of possible z is $1/2$
 - sum of possible z is 1
 - product of possible z is $1/4$
 - product of possible z is $1/2$

Multiple Correct Answers Type

1. If $z = \omega, \omega^2$, where ω is a nonreal complex cube root of unity, are two vertices of an equilateral triangle in the Argand plane, then the third vertex may be represented by
- $z = 1$
 - $z = 0$
 - $z = -2$
 - $z = -1$
2. If $\text{amp}(z_1 z_2) = 0$ and $|z_1| = |z_2| = 1$, then
- $z_1 + z_2 = 0$
 - $z_1 z_2 = 1$
 - $z_1 = \bar{z}_2$
 - none of these
3. If $\sqrt{5 - 12i} + \sqrt{-5 - 12i} = z$, then principal value of $\arg z$ can be
- $-\frac{\pi}{4}$
 - $\frac{\pi}{4}$
 - $\frac{3\pi}{4}$
 - $-\frac{3\pi}{4}$
4. Value(s) $(-i)^{1/3}$ is/are
- $\frac{\sqrt{3} - i}{2}$
 - $\frac{\sqrt{3} + i}{2}$
 - $\frac{-\sqrt{3} - i}{2}$
 - $\frac{-\sqrt{3} + i}{2}$

13. If $|z - 3| = \min\{|z - 1|, |z - 5|\}$, then $\operatorname{Re}(z)$ equals to
 (1) 2 (2) $\frac{5}{2}$ (3) $\frac{7}{2}$ (4) 4
14. If z_1, z_2 are two complex numbers ($z_1 \neq z_2$) satisfying $|z_1^2 - z_2^2| = |\bar{z}_1^2 + \bar{z}_2^2 - 2\bar{z}_1\bar{z}_2|$, then
 (1) $\frac{z_1}{z_2}$ is purely imaginary (2) $\frac{z_1}{z_2}$ is purely real
 (3) $|\arg z_1 - \arg z_2| = \pi$ (4) $|\arg z_1 - \arg z_2| = \frac{\pi}{2}$
15. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\operatorname{Re}(z_1\bar{z}_2) = 0$, then the pair of complex numbers $\omega_1 = a + ic$ and $\omega_2 = b + id$ satisfies
 (1) $|\omega_1| = 1$ (2) $|\omega_2| = 1$
 (3) $\operatorname{Re}(\omega_1\bar{\omega}_2) = 0$ (4) $\operatorname{Im}(\omega_1\bar{\omega}_2) = 0$
16. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\frac{z_1 + z_2}{z_1 - z_2}$ may be
 (1) zero (2) real and positive
 (3) real and negative (4) purely imaginary
17. If $|z_1| = \sqrt{2}$, $|z_2| = \sqrt{3}$ and $|z_1 + z_2| = \sqrt{5 - 2\sqrt{3}}$, then $\arg\left(\frac{z_1}{z_2}\right)$ (not necessarily principal) is
 (1) $\frac{3\pi}{4}$ (2) $\frac{2\pi}{3}$ (3) $\frac{5\pi}{4}$ (4) $\frac{\pi}{3}$
18. Let four points z_1, z_2, z_3, z_4 be in complex plane such that $|z_1| < 1$, $|z_2| = 1$ and $|z_3| \leq 1$. If $z_3 = \frac{z_2(z_1 - z_4)}{\bar{z}_1 z_4 - 1}$, then $|z_4|$ can be
 (1) 2 (2) $\frac{2}{5}$ (3) $\frac{1}{3}$ (4) $\frac{5}{2}$
19. A rectangle of maximum area is inscribed in the circle $|z - 3 - 4i| = 1$. If one vertex of the rectangle is $4 + 4i$, then another adjacent vertex of this rectangle can be
 (1) $2 + 4i$ (2) $3 + 5i$ (3) $3 + 3i$ (4) $3 - 3i$
20. If $|z_1| = 15$ and $|z_2 - 3 - 4i| = 5$, then
 (1) $|z_1 - z_2|_{\min} = 5$ (2) $|z_1 - z_2|_{\min} = 10$
 (3) $|z_1 - z_2|_{\max} = 20$ (4) $|z_1 - z_2|_{\max} = 25$
21. $P(z_1)$, $Q(z_2)$, $R(z_3)$, and $S(z_4)$ are four complex numbers representing the vertices of a rhombus taken in order on the complex plane, then which one of the following is/are correct?
 (1) $\frac{z_1 - z_4}{z_2 - z_3}$ is purely real
 (2) $\operatorname{amp} \frac{z_1 - z_4}{z_2 - z_4} = \operatorname{amp} \frac{z_2 - z_4}{z_3 - z_4}$
 (3) $\frac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary
 (4) it is not necessary that $|z_1 - z_3| \neq |z_2 - z_4|$
22. If $\arg(z + a) = \pi/6$ and $\arg(z - a) = 2\pi/3$ ($a \in \mathbb{R}^+$), then
 (1) $|z| = a$ (2) $|z| = 2a$
 (3) $\arg(z) = \frac{\pi}{2}$ (4) $\arg(z) = \frac{\pi}{3}$
23. If a complex number z satisfies $|z| = 1$ and $\arg(z - 1) = \frac{2\pi}{3}$, then (ω is complex cube root of unity)
 (1) $z^2 + z$ is purely imaginary number
 (2) $z = -\omega^2$
 (3) $z = -\omega$
 (4) $|z - 1| = 1$
24. If $|z - 1| = 1$, then
 (1) $\arg((z - 1 - i)/z)$ can be equal to $-\pi/4$
 (2) $(z - 2)/z$ is purely imaginary number
 (3) $(z - 2)/z$ is purely real number
 (4) if $\arg(z) = \theta$, where $z \neq 0$ and θ is acute, then $1 - 2/z = i \tan \theta$
25. If $z_1 = 5 + 12i$ and $|z_2| = 4$, then
 (1) maximum $(|z_1 + iz_2|) = 17$
 (2) minimum $(|z_1 + (1 + i)z_2|) = 13 - 4\sqrt{2}$
 (3) minimum $\left| \frac{z_1}{z_2 + \frac{4}{z_2}} \right| = \frac{13}{4}$
 (4) maximum $\left| \frac{z_1}{z_2 + \frac{4}{z_2}} \right| = \frac{13}{3}$
26. Let z_1, z_2, z_3 be the three nonzero complex numbers such that $z_2 \neq 1$, $a = |z_1|$, $b = |z_2|$ and $c = |z_3|$. Let $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$ Then
 (1) $\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2$
 (2) orthocenter of triangle formed by z_1, z_2, z_3 is $z_1 + z_2 + z_3$
 (3) if triangle formed by z_1, z_2, z_3 is equilateral, then its area is $\frac{3\sqrt{3}}{2}|z_1|^2$
 (4) if triangle formed by z_1, z_2, z_3 is equilateral, then $z_1 + z_2 + z_3 = 0$
27. z_1 and z_2 are the roots of the equation $z^2 - az + b = 0$, where $|z_1| = |z_2| = 1$ and a, b are nonzero complex numbers, then
 (1) $|a| \leq 1$ (2) $|a| \leq 2$
 (3) $\arg(a^2) = \arg(b)$ (4) $\arg a = \arg(b^2)$
28. If $|(z - z_1)/(z - z_2)| = 3$, where z_1 and z_2 are fixed complex numbers and z is a variable complex number, then z lies on a
 (1) circle with z_1 as its interior point
 (2) circle with z_2 as its interior point
 (3) circle with z_1 as its exterior point
 (4) circle with z_2 as its exterior point

29. If $z = x + iy$, then the equation $|(2z - i)/(z + 1)| = m$ represents a circle, then m can be
 (1) $1/2$ (2) 1
 (3) 2 (4) 3
30. System of equations $|z + 3| - |z - 3| = 6$ and $|z - 4| = r$, where $r \in R^+$ has
 (1) one solution if $r > 1$ (2) one solution if $r < 1$
 (3) two solutions if $r = 1$ (4) at least one solution
31. Let the equation of a ray be $|z - 2| - |z - 1 - i| = \sqrt{2}$. If it strikes the y -axis, then the equation of reflected ray (including or excluding the point of incidence) is
 (1) $\arg(z - 2i) = \frac{\pi}{4}$ (2) $|z - 2i| - |z - 1 - 3i| = \sqrt{2}$
 (3) $\arg(z - 2i) = \frac{3\pi}{4}$ (4) $|z - 2i| - |z - 1 - 3i| = 2\sqrt{2}$
32. Given that the two curves $\arg(z) = \pi/6$ and $|z - 2\sqrt{3}i| = r$ intersect in two distinct points, then
 (1) $[r] \neq 2$, where $[.]$ represents greatest integer
 (2) $0 < r < 3$
 (3) $r = 6$
 (4) $3 < r < 2\sqrt{3}$
 ($[r]$ represents integral part of r)
33. On the Argand plane, let $z_1 = -2 + 3iz$, $z_2 = -2 - 3iz$ and $|z| = 1$. Then
 (1) z_1 moves on circle with centre at $(-2, 0)$ and radius 3
 (2) z_1 and z_2 describe the same locus
 (3) z_1 and z_2 move on different circles
 (4) $z_1 - z_2$ moves on a circle concentric with $|z| = 1$
34. Let $S = \{z : x = x + iy, y \geq 0, |z - z_0| \leq 1\}$, where $|z_0| = |z_0 - \omega| = |z_0 - \omega^2|$, ω and ω^2 are non-real cube roots of unity. Then
 (1) $z_0 = -1$
 (2) $z_0 = -1/2$
 (3) if $z \in S$, then least value of $|z|$ is 1
 (4) $|\arg(\omega - z_0)| = \pi/3$
35. If P and Q are represented by the complex numbers z_1 and z_2 , such that $|1/z_2 + 1/z_1| = |1/z_2 - 1/z_1|$, then
 (1) $\triangle OPQ$ (where O is the origin) is equilateral
 (2) $\triangle OPQ$ is right angled
 (3) the circumcenter of $\triangle OPQ$ is $\frac{1}{2}(z_1 + z_2)$
 (4) the circumcenter of $\triangle OPQ$ is $\frac{1}{3}(z_1 + z_2)$
36. Locus of complex number satisfying $\arg[(z - 5 + 4i)/(z + 3 - 2i)] = -\pi/4$ is the arc of a circle
 (1) whose radius is $5\sqrt{2}$
 (2) whose radius is 5
 (3) whose length (of arc) is $\frac{15\pi}{\sqrt{2}}$
 (4) whose centre is $-2 - 5i$
37. Equation of tangent drawn to circle $|z| = r$ at the point $A(z_0)$ is
 (1) $\operatorname{Re}\left(\frac{z}{z_0}\right) = 1$ (2) $z\bar{z}_0 + z_0\bar{z} = 2r^2$
 (3) $\operatorname{Im}\left(\frac{z}{z_0}\right) = 1$ (4) $\operatorname{Im}\left(\frac{z_0}{z}\right) = 1$
38. If n is a natural number ≥ 2 , such that $z^n = (z + 1)^n$, then
 (1) roots of equation lie on a straight line parallel to the y -axis
 (2) roots of equation lie on a straight line parallel to the x -axis
 (3) sum of the real parts of the roots is $-(n-1)/2$
 (4) none of these
39. If $|z - (1/z)| = 1$, then
 (1) $|z|_{\max} = \frac{1+\sqrt{5}}{2}$ (2) $|z|_{\min} = \frac{\sqrt{5}-1}{2}$
 (3) $|z|_{\max} = \frac{\sqrt{5}-2}{2}$ (4) $|z|_{\min} = \frac{\sqrt{5}-1}{\sqrt{2}}$
40. If $1, z_1, z_2, z_3, \dots, z_{n-1}$ be the n th roots of unity and ω be a non-real complex cube root of unity, then the product $\prod_{r=1}^{n-1} (\omega - z_r)$ can be equal to
 (1) 0 (2) 1 (3) -1 (4) $1 + \omega$
41. Let z be a complex number satisfying equation $z^p = \bar{z}^q$, where $p, q \in N$, then
 (1) if $p = q$, then number of solutions of equation will be infinite
 (2) if $p = q$, then number of solutions of equation will be finite
 (3) if $p \neq q$, then number of solutions of equation will be $p + q + 1$.
 (4) if $p \neq q$, then number of solutions of equation will be $p + q$
42. Which of the following is true?
 (1) The number of common roots of $z^{144} = 1$ and $z^{24} = 1$ is 24
 (2) The number of common roots of $z^{360} = 1$ and $z^{315} = 1$ is 45
 (3) The number of roots common to $z^{24} = 1$, $z^{20} = 1$ and $z^{56} = 1$ is 4
 (4) The number of roots common to $z^{27} = 1$, $z^{125} = 1$ and $z^{49} = 1$ is 1
43. If from a point P representing the complex number z_1 on the curve $|z| = 2$, two tangents are drawn from P to the curve $|z| = 1$, meeting at points $Q(z_2)$ and $R(z_3)$, then
 (1) complex number $(z_1 + z_2 + z_3)/3$ will be on the curve $|z| = 1$
 (2) $\left(\frac{4}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3}\right)\left(\frac{4}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) = 9$
 (3) $\arg\left(\frac{z_2}{z_3}\right) = \frac{2\pi}{3}$
 (4) orthocenter and circumcenter of $\triangle PQR$ will coincide

44. A complex number z is rotated in anticlockwise direction by an angle α and we get z' and if the same complex number z is rotated by an angle α in clockwise direction and we get z'' , then

- (1) z', z, z'' are in G.P. (2) z', z, z'' are in H.P.
 (3) $z' + z'' = 2z \cos \alpha$ (4) $z'^2 + z''^2 = 2z^2 \cos 2\alpha$

45. z_1, z_2, z_3 and z'_1, z'_2, z'_3 are nonzero complex numbers such that $z_3 = (1 - \lambda)z_1 + \lambda z_2$ and $z'_3 = (1 - \mu)z'_1 + \mu z'_2$, then which of the following statements is/are true?

- (1) If $\lambda, \mu \in \mathbb{R} - \{0\}$, then z_1, z_2 , and z_3 are collinear and z'_1, z'_2, z'_3 are collinear separately.
 (2) If λ, μ are complex numbers, where $\lambda = \mu$, then triangles formed by points z_1, z_2, z_3 and z'_1, z'_2, z'_3 are similar.
 (3) If λ, μ are distinct complex numbers, then points z_1, z_2, z_3 and z'_1, z'_2, z'_3 are not connected by any well defined geometry.
 (4) If $0 < \lambda < 1$, then z_3 divides the line joining z_1 and z_2 internally and if $\mu > 1$, then z'_3 divides the line joining of z'_1, z'_2 externally.

46. Given $z = f(x) + i g(x)$ where $f, g: (0, 1) \rightarrow (0, 1)$ are real valued functions. Then which of the following does not hold good?

- (1) $z = \frac{1}{1 - ix} + i \left(\frac{1}{1 + ix} \right)$
 (2) $z = \frac{1}{1 + ix} + i \left(\frac{1}{1 - ix} \right)$
 (3) $z = \frac{1}{1 + ix} + i \left(\frac{1}{1 + ix} \right)$
 (4) $z = \frac{1}{1 - ix} + i \left(\frac{1}{1 - ix} \right)$

47. Consider three distinct complex numbers a, b and c such that $|a| = |b| = |c| = 1$. Also, z_1 and z_2 are the roots of the equation $az^2 + bz + c = 0$ with $|z_1| = 1$. If P and Q represent the complex numbers z_1 and z_2 in the Argand plane with $\angle POQ = \theta$, $0 < \theta < 180^\circ$ (where O being the origin), then

- (1) $b^2 = ac$ (2) $PQ = \sqrt{3}$
 (3) $\theta = \frac{\pi}{3}$ (4) $\theta = \frac{2\pi}{3}$

48. If all the three roots of $az^3 + bz^2 + cz + d = 0$ have negative real parts ($a, b, c \in \mathbb{R}$), then

- (1) $ab > 0$ (2) $bc > 0$
 (3) $ad > 0$ (4) $bc - ad > 0$

49. If $\frac{3}{2 + e^{i\theta}} = ax + iby$, then the locus of $P(x, y)$ will represent

- (1) ellipse if $a = 1, b = 2$
 (2) circle if $a = b = 1$
 (3) pair of straight line if $a = 1, b = 0$
 (4) None of these

Linked Comprehension Type

For Problems 1–4

Consider the complex numbers $z = (1 - i \sin \theta)/(1 + i \cos \theta)$.

1. The value of θ for which z is purely real are

- (1) $n\pi - \frac{\pi}{4}, n \in I$ (2) $n\pi + \frac{\pi}{4}, n \in I$
 (3) $n\pi, n \in I$ (4) none of these

2. The value of θ for which z is purely imaginary are

- (1) $n\pi - \frac{\pi}{4}, n \in I$ (2) $n\pi + \frac{\pi}{4}, n \in I$
 (3) $n\pi, n \in I$ (4) no real values of θ

3. The value of θ for which z is unimodular is given by

- (1) $n\pi \pm \frac{\pi}{6}, n \in I$ (2) $n\pi \pm \frac{\pi}{3}, n \in I$
 (3) $n\pi \pm \frac{\pi}{4}, n \in I$ (4) no real values of θ

4. If argument of z is $\pi/4$, then

- (1) $\theta = n\pi, n \in I$ only
 (2) $\theta = (2n + 1)\pi, n \in I$ only
 (3) both $\theta = n\pi$ and $\theta = (2n + 1)\frac{\pi}{2}, n \in I$
 (4) none of these

For Problems 5–8

Consider the complex numbers z_1 and z_2 satisfying the relation $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$.

5. Complex number $z_1 \bar{z}_2$ is

- (1) purely real (2) purely imaginary
 (3) zero (4) none of these

6. Complex number z_1/z_2 is

- (1) purely real (2) purely imaginary
 (3) zero (4) none of these

7. One of the possible argument of complex number $i(z_1/z_2)$

- (1) $\frac{\pi}{2}$ (2) $-\frac{\pi}{2}$
 (3) 0 (4) none of these

8. Possible difference between the argument of z_1 and z_2 is

- (1) 0 (2) π
 (3) $-\frac{\pi}{2}$ (4) none of these

For Problems 9–11

Let z be a complex number satisfying $z^2 + 2z\lambda + 1 = 0$, where λ is a parameter which can take any real value.

9. The roots of this equation lie on a certain circle if

- (1) $-1 < \lambda < 1$ (2) $\lambda > 1$
 (3) $\lambda < 1$ (4) none of these

10. One root lies inside the unit circle and one outside if
 (1) $-1 < \lambda < 1$ (2) $\lambda > 1$
 (3) $\lambda < 1$ (4) none of these
11. For every large value of λ , the roots are approximately
 (1) $-2\lambda, 1/\lambda$ (2) $-\lambda, -1/\lambda$
 (3) $-2\lambda, -\frac{1}{2\lambda}$ (4) none of these

For Problems 12 and 13

The roots of the equation $z^4 + az^3 + (12 + 9i)z^2 + bz = 0$ (where a and b are complex numbers) are the vertices of a square. Then

12. The value of $|a - b|$ is
 (1) $5\sqrt{5}$ (2) $\sqrt{130}$ (3) 12 (4) $\sqrt{175}$
13. The area of the square is
 (1) 25 sq. units (2) 20 sq. units
 (3) 5 sq. unit (4) 4 sq. units

For Problems 14–16

Consider a quadratic equation $az^2 + bz + c = 0$, where a, b, c are complex numbers.

14. The condition that the equation has one purely imaginary root is
 (1) $(c\bar{a} - a\bar{c})^2 = -(b\bar{c} + c\bar{b})(a\bar{b} + \bar{a}b)$
 (2) $(c\bar{a} + a\bar{c})^2 = (b\bar{c} + c\bar{b})(a\bar{b} + \bar{a}b)$
 (3) $(c\bar{a} - a\bar{c})^2 = (b\bar{c} - c\bar{b})(a\bar{b} - \bar{a}b)$
 (4) none of these
15. If equation has two purely imaginary roots, then which of the following is not true.
 (1) $a\bar{b}$ is purely imaginary (2) $b\bar{c}$ is purely imaginary
 (3) $c\bar{a}$ is purely real (4) none of these
16. The condition that the equation has one purely real root is
 (1) $(c\bar{a} - a\bar{c})^2 = (b\bar{c} + c\bar{b})(a\bar{b} - \bar{a}b)$
 (2) $(c\bar{a} - a\bar{c})^2 = (b\bar{c} - c\bar{b})(a\bar{b} + \bar{a}b)$
 (3) $(c\bar{a} - a\bar{c})^2 = (b\bar{c} + c\bar{b})(a\bar{b} + \bar{a}b)$
 (4) $(c\bar{a} - a\bar{c})^2 = (b\bar{c} - c\bar{b})(a\bar{b} - \bar{a}b)$

For Problems 17–19

Suppose z and ω are two complex numbers such that

$$|z| \leq 1, |\omega| \leq 1, \text{ and } |z + i\omega| = |z - i\omega| = 2$$

17. Which of the following is true about $|z|$ and $|\omega|$?
 (1) $|z| = |\omega| = \frac{1}{2}$ (2) $|z| = \frac{1}{2}, |\omega| = \frac{3}{4}$
 (3) $|z| = |\omega| = \frac{3}{4}$ (4) $|z| = |\omega| = 1$
18. Which of the following is true for z and ω ?
 (1) $\operatorname{Re}(z) = \operatorname{Re}(\omega)$ (2) $\operatorname{Im}(z) = \operatorname{Im}(\omega)$
 (3) $\operatorname{Re}(z) = \operatorname{Im}(\omega)$ (4) $\operatorname{Im}(z) = \operatorname{Re}(\omega)$
19. The complex number ω can be
 (1) 1 or $-i$
 (2) -1
 (3) i or $-i$
 (4) ω or ω^2 (where ω is the cube root of unity)

For Problems 20–22

Consider the equation of line $a\bar{z} + \bar{a}z + b = 0$, where b is a real parameter and a is fixed non-zero complex number.

20. The intercept of line on real axis is given by
 (1) $\frac{-2b}{a + \bar{a}}$ (2) $\frac{-b}{2(a + \bar{a})}$ (3) $\frac{-b}{a + \bar{a}}$ (4) $\frac{b}{a + \bar{a}}$
21. The intercept of line on imaginary axis is given by
 (1) $\frac{b}{\bar{a} - a}$ (2) $\frac{2b}{\bar{a} - a}$ (3) $\frac{b}{2(\bar{a} - a)}$ (4) $\frac{b}{a - \bar{a}}$
22. The locus of mid-point of the line intercepted between real and imaginary axis is given by
 (1) $az - \bar{a}\bar{z} = 0$ (2) $az + \bar{a}\bar{z} = 0$
 (3) $az - \bar{a}\bar{z} + b = 0$ (4) $az - \bar{a}\bar{z} + 2b = 0$

For Problems 23–25

Consider the equation $az + b\bar{z} + c = 0$, where $a, b, c \in \mathbb{C}$.

23. If $|a| \neq |b|$, then z represents
 (1) circle (2) straight line
 (3) one point (4) ellipse
24. If $|a| = |b|$ and $\bar{a}c \neq b\bar{c}$, then z has
 (1) infinite solutions (2) no solutions
 (3) finite solutions (4) cannot say anything
25. If $|a| = |b| \neq 0$ and $\bar{a}c = b\bar{c}$, then $az + b\bar{z} + c = 0$ represents
 (1) an ellipse (2) a circle
 (3) a point (4) a straight line

For Problems 26–28

Complex numbers z satisfy the equation $|z - (4/z)| = 2$.

26. The difference between the least and the greatest moduli of complex numbers is
 (1) 2 (2) 4 (3) 1 (4) 3
27. The value of $\arg(z_1/z_2)$, where z_1 and z_2 are complex numbers with the greatest and the least moduli, can be
 (1) 2π (2) π
 (3) $\pi/2$ (4) none of these
28. Locus of z if $|z - z_1| = |z - z_2|$, where z_1 and z_2 are complex numbers with the greatest and the least moduli, is
 (1) line parallel to the real axis
 (2) line parallel to the imaginary axis
 (3) line having a positive slope
 (4) line having a negative slope

For Problems 29–31

In an Argand plane z_1, z_2 , and z_3 are, respectively, the vertices of an isosceles triangle ABC with $AC = BC$ and $\angle CAB = \theta$. If z_4 is incenter of triangle, then

29. the value of $AB \times AC / (IA)^2$ is
 (1) $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2}$ (2) $\frac{(z_2 - z_1)(z_1 - z_3)}{(z_4 - z_1)^2}$
 (3) $\frac{(z_4 - z_1)}{(z_2 - z_1)(z_3 - z_1)}$ (4) none of these

30. the value of $(z_4 - z_1)^2 (\cos \theta + 1) \sec \theta$ is

- (1) $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)}$
- (2) $(z_2 - z_1)(z_3 - z_1)$
- (3) $(z_2 - z_1)(z_3 - z_1)^2$
- (4) $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2}$

31. the value of $(z_2 - z_1)^2 \tan \theta \tan \theta/2$ is

- (1) $(z_1 + z_2 - 2z_3)$
- (2) $(z_1 + z_2 - z_3)(z_1 + z_2 - z_4)$
- (3) $-(z_1 + z_2 - 2z_3)(z_1 + z_2 - 2z_4)$
- (4) none of these

For Problems 32–34

$A(z_1), B(z_2), C(z_3)$ are the vertices of a triangle ABC inscribed in the circle $|z| = 2$. Internal angle bisector of the angle A meets the circumcircle again at $D(z_4)$.

32. Complex number representing point D is

- (1) $z_4 = \frac{1}{z_2} + \frac{1}{z_3}$
- (2) $\sqrt{\frac{z_2 + z_3}{z_1}}$
- (3) $\sqrt{\frac{z_2 z_3}{z_1}}$
- (4) $z_4 = \sqrt{z_2 z_3}$

33. $\arg [z_4/(z_2 - z_3)]$ is equal to

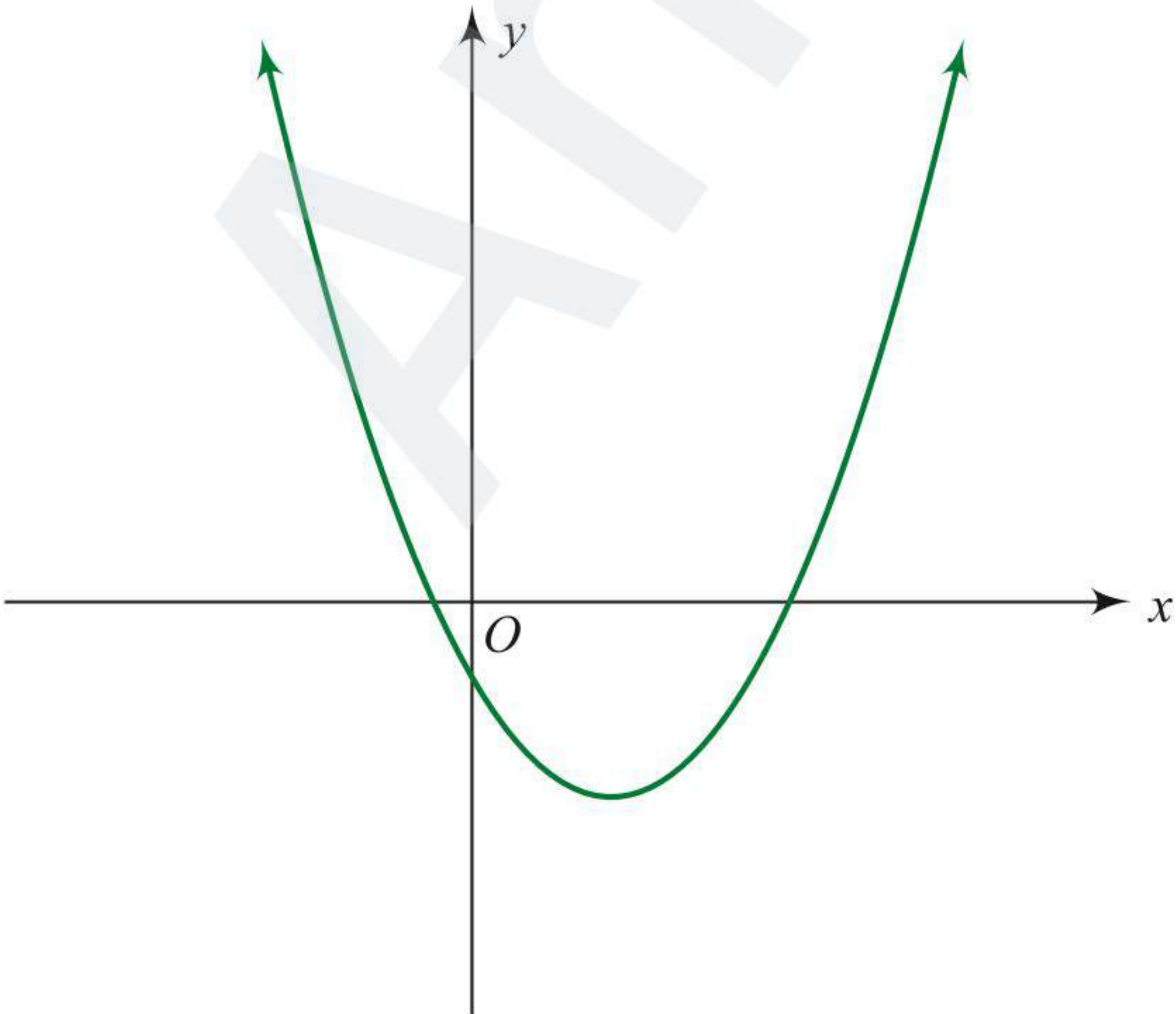
- (1) $\frac{\pi}{4}$
- (2) $\frac{\pi}{3}$
- (3) $\frac{\pi}{2}$
- (4) $\frac{2\pi}{3}$

34. For fixed positions of $B(z_2)$ and $C(z_3)$ all the bisectors (internal) of $\angle A$ will pass through a fixed point which is

- (1) H.M. of z_2 and z_3
- (2) A.M. of z_2 and z_3
- (3) G.M. of z_2 and z_3
- (4) none of these

Matrix Match Type

1. The graph of the quadratic function $y = ax^2 + bx + c$ is as shown in the following figure.



Now, match the complex numbers given in List I with the corresponding arguments in List II.

List I	List II
a. $a + ib$	p. $-\tan^{-1} \frac{b}{a}$
b. $ac - ibc$	q. $\tan^{-1} \frac{b}{a}$
c. $-ac + ibc$	r. $\pi + \tan^{-1} \frac{b}{a}$
d. $abc - ib^2c$	s. $-\pi - \tan^{-1} \frac{b}{a}$

2. Let z_1, z_2 and z_3 be the vertices of a triangle. Then match following lists:

List I	List II
a. If $2(z_3 - z_1) + (i\sqrt{3} + 1)(z_1 - z_2) = 0$ then triangle is	p. right angled but may not be isosceles
b. $\operatorname{Re}\left(\frac{z_3 - z_1}{z_3 - z_2}\right) = 0$, then triangle is	q. obtuse angled
c. $\operatorname{Re}\left(\frac{z_3 - z_1}{z_3 - z_2}\right) < 0$, then triangle is	r. isosceles and right angled
d. $\frac{z_3 - z_1}{z_3 - z_2} = i$, then triangle is	s. equilateral

3. Match the following lists:

List I	List II
a. For complex number z , the minimum value of $ z + z - \cos \alpha - i(\sin \alpha) $ is	p. 0
b. Number of complex numbers z satisfying $ z + 2 + z - 2 = 8$ and $ z - 1 + z + 1 = 2$ is	q. 1
c. If $\left \frac{z_1 - 3z_2}{3 - z_1 \bar{z}_2}\right = 1$ and $ z_2 \neq 1$, then $ z_1 $ is equal to	r. 2
d. Let $z = \frac{(2\sqrt{3} + 2i)^8}{(1 - i)^6} + \frac{(1 + i)^6}{(2\sqrt{3} - 2i)^8}$. If θ is argument of z such that $\theta \in (-\pi, \pi]$ then $ 4\sin \theta $ is equal to	s. 3

4. Complex number z satisfies the equation $||z - 5| + m|z - 12i|| = n$. Then match the values of m and n in List I with the corresponding locus in List II.

List I	List II
a. $m = 1, n = 13$	p. Ellipse
b. $m = 1, n = 15$	q. Hyperbola
c. $m = -1, n = 13$	r. Line segment
d. $m = -1, n = 12$	s. Two opposite rays

5. Complex number z lies on the curve $S \equiv \arg \frac{z+3}{z+3i} = -\frac{\pi}{4}$.

Now, match the locus in List I with its number of points of intersection with the curve S in List II.

List I	List II
a. $ z+3 - z+3i = 3\sqrt{2}$	p. 0
b. $ z-1-i + z-4-5i = 5$	q. 1
c. $\arg(z-i) = -\frac{3\pi}{4}$	r. 2
d. $\arg(z+3-4i) = -\frac{\pi}{4}$	s. 4

Codes:

- | | | | | |
|-----|---|---|---|---|
| | a | b | c | d |
| (1) | p | q | p | r |
| (2) | s | r | q | p |
| (3) | q | p | q | r |
| (4) | s | p | q | r |

6. Consider sets $A = \{z \in C : z^{27} - 1 = 0\}$ and $B = \{z \in C : z^{36} - 1 = 0\}$. Now, match the following lists.

List I	List II
a. Number of elements in A which have negative imaginary part is	p. 18
b. Number of elements in B which have positive real part is	q. 17
c. Number of elements in $A \cap B$ is	r. 13
d. If number of elements in $A \cup B$ is p then p is divisible by	s. 9

Codes:

- | | | | | |
|-----|---|---|---|---|
| | a | b | c | d |
| (1) | p | q | p | r |
| (2) | r | q | s | p |
| (3) | q | p | q | r |
| (4) | s | p | q | r |

Numerical Value Type

- If $x = a + bi$ is a complex number such that $x^2 = 3 + 4i$ and $x^3 = 2 + 11i$, where $i = \sqrt{-1}$ then $(a+b)$ equal to _____.
- If the complex numbers x and y satisfy $x^3 - y^3 = 98i$ and $x - y = 7i$, then $xy = a + ib$, where $a, b \in R$. The value of $(a+b)$ equals _____.
- If $x = \omega - \omega^2 - 2$, then the value of $x^4 + 3x^3 + 2x^2 - 11x - 6$ is (where ω is cube root of unity) _____.
- Let $z = 9 + bi$, where b is nonzero real and $i^2 = -1$. If the imaginary part of z^2 and z^3 are equal, then b is _____.
- Modulus of nonzero complex number z satisfying $\bar{z} + z = 0$ and $|z|^2 - 4zi = z^2$ is _____.
- If the expression $(1 + ir)^3$ is of the form of $s(1 + i)$ for some real s where r is also real and, then the sum of all possible values of r is _____.

- If complex number $z (z \neq 2)$ satisfies the equation $z^2 = 4z + |z|^2 + \frac{16}{|z|^3}$, then the value of $|z|^4$ is _____.
- The complex number z satisfies $z + |z| = 2 + 8i$. The value of $(|z|)$ is _____.
- Let $|z| = 2$ and $w = \frac{z+1}{z-1}$, where $z, w \in C$ (where C is the set of complex numbers). Then product of least and greatest value of modulus of w is _____.
- If z is a complex number satisfying $z^4 + z^3 + 2z^2 + z + 1 = 0$, then $|z|$ is equal to _____.
- Let $1, \omega, \omega^2$ be the cube roots of unity. The least possible degree of a polynomial with real coefficients having roots $2\omega, (2+3\omega), (2+3\omega^2), (2-\omega-\omega^2)$ is _____.
- If ω is the imaginary cube root of unity, then the number of pairs of integers (a, b) such that $|a\omega + b| = 1$ is _____.
- Suppose that z is a complex number that satisfies $|z-2-2i| \leq 1$. The maximum value of $|2iz + 4|$ is equal to _____.
- If $|z+2-i| = 5$ and maximum value of $|3z+9-7i|$ is M , then the value of M is _____.
- Let $Z_1 = (8+i)\sin\theta + (7+4i)\cos\theta$ and $Z_2 = (1+8i)\sin\theta + (4+7i)\cos\theta$ are two complex numbers. If $Z_1 \cdot Z_2 = a + ib$, where $a, b \in R$. If M is the greatest value of $(a+b) \forall \theta \in R$, then the value of M is _____.
- Let $A = \{a \in R \mid \text{the equation } (1+2i)x^3 - 2(3+i)x^2 + (5-4i)x + 2a^2 = 0 \text{ has at least one real root. Then the value of } \sum a^2 \text{ is } ______.$
- The minimum value of the expression $E = |z|^2 + |z-3|^2 + |z-6i|^2$ is m , then the value of m is _____.
- If z_1 lies on $|z-3| + |z+3| = 8$ such that $\arg z_1 = \pi/6$, then $37|z_1|^2 = ______.$
- If z satisfies the condition $\arg(z+i) = \frac{\pi}{4}$. Then the minimum value of $|z+1-i| + |z-2+3i|$ is _____.
- Let $\omega \neq 1$ be a complex cube root of unity. If $(4+5\omega+6\omega^2)^{n^2+2} + (6+5\omega^2+4\omega)^{n^2+2} + (5+6\omega+4\omega^2)^{n^2+2} = 0$, and $n \in N$; where $n \in [1, 100]$, then number of values of n is _____.
- Let z be a non-real complex number which satisfies the equation $z^{23} = 1$. Then the value of $\sum_{k=1}^{22} \frac{1}{1+z^{8k}+z^{16k}}$ is _____.
- If z, z_1 and z_2 are complex numbers such that $z = z_1 z_2$ and $|\bar{z}_2 - z_1| \leq 1$, then maximum value of $|z| - \operatorname{Re}(z)$ is _____.
- Let z_1, z_2 and z_3 be three complex numbers such that $z_1 + z_2 + z_3 = z_1 z_2 + z_2 z_3 + z_1 z_3 = z_1 z_2 z_3 = 1$. Then the area of triangle formed by points $A(z_1), B(z_2)$ and $C(z_3)$ in complex plane is _____.

24. Let α be the non-real 5th root of unity. If z_1 and z_2 are two complex numbers lying on $|z| = 2$, then the value of $\sum_{t=0}^4 |z_1 + \alpha^t z_2|^2$ is _____.
25. Let $z_1, z_2, z_3 \in C$ such that $|z_1| = |z_2| = |z_3| = |z_1 + z_2 + z_3| = 4$. If $|z_1 - z_2| = |z_1 - z_3|$ and $z_2 \neq z_3$, then the value of $|z_1 + z_2| \cdot |z_1 + z_3|$ is _____.
26. Let $A(z_1)$ and $B(z_2)$ be lying on the curve $|z - 3 - 4i| = 5$, where $|z_1|$ is maximum. Now, $A(z_1)$ is rotated about the origin in anticlockwise direction through 90° reaching

at $P(z_0)$. If A , B and P are collinear then the value of $(|z_0 - z_1| \cdot |z_0 - z_2|)$ is _____.

27. If z_1, z_2 and z_3 are three points lying on the circle $|z| = 2$, then minimum value of $|z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2$ is _____.
28. Minimum value of $|z_1 + 1| + |z_2 + 1| + |z_1 z_2 + 1|$ if $|z_1| = 1$ and $|z_2| = 1$ is _____.
29. If $|z_1| = 2$ and $(1 - i)z_2 + (1 + i)\bar{z}_2 = 8\sqrt{2}$, then the minimum value of $|z_1 - z_2|$ is _____.
30. Given that $1 + 2|z|^2 = |z^2 + 1|^2 + 2|z + 1|^2$, then the value of $|z(z + 1)|$ is _____.

Archives

JEE ADVANCED

Single Correct Answer Type

1. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $\bar{z}z^3 + z\bar{z}^3 = 350$ is
(1) 48 (2) 32 (3) 40 (4) 80

(IIT-JEE, 2009)

2. Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a cannot take the value

- (1) -1 (2) $\frac{1}{3}$ (3) $\frac{1}{2}$ (4) $\frac{3}{4}$

(IIT-JEE, 2012)

3. Let complex numbers α and $\frac{1}{\alpha}$ lie on circles $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$
(1) $1/\sqrt{2}$ (2) $1/2$ (3) $1/\sqrt{7}$ (4) $1/3$

(JEE Advanced 2013)

4. Let S be the set of all complex numbers z satisfying $|z - 2 + i| \geq \sqrt{5}$. If the complex number z_0 is such that

$\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{ \frac{1}{|z - 1|} : z \in S \right\}$, then

the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is

- (1) $\frac{\pi}{4}$ (2) $\frac{3\pi}{4}$
(3) $-\frac{\pi}{2}$ (4) $\frac{\pi}{2}$

(JEE Advanced 2019)

5. Let $\theta_1, \theta_2, \dots, \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + \dots + \theta_{10} = 2\pi$. Define the complex numbers

$z_1 = e^{i\theta_1}$, $z_k = z_{k-1}e^{i\theta_k}$ for $k = 2, 3, \dots, 10$, where $i = \sqrt{-1}$.

Consider the statements P and Q given below:

$P : |z_2 - z_1| + |z_3 - z_2| + \dots + |z_{10} - z_9| + |z_1 - z_{10}| \leq 2\pi$

$Q : |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + \dots + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \leq 4\pi$

Then,

- (1) P is TRUE and Q is FALSE
(2) Q is TRUE and P is FALSE
(3) both P and Q are TRUE
(4) both P and Q are FALSE

(JEE Advanced 2021)

Multiple Correct Answers Type

1. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If $\arg(w)$ denotes the principal argument of a nonzero complex number w , then

- (1) $|z - z_1| + |z - z_2| = |z_1 - z_2|$
(2) $(z - z_1) = (z - z_2)$

- (3) $\left| \frac{z - z_1}{z_2 - z_1} \cdot \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \right| = 0$

- (4) $\arg(z - z_1) = \arg(z_2 - z_1)$

(IIT-JEE, 2010)

2. Let $w = \frac{\sqrt{3} + i}{2}$ and $P = \{w^n : n = 1, 2, 3, \dots\}$. Further

$H_1 = \left\{ z \in C : \operatorname{Re} z > \frac{1}{2} \right\}$ and $H_2 = \left\{ z \in C : \operatorname{Re} z < \frac{-1}{2} \right\}$,

where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$, and O represents the origin, then $\angle z_1 O z_2 =$

- (1) $\pi/2$ (2) $\pi/6$ (3) $2\pi/3$ (4) $5\pi/6$

(JEE Advanced 2013)

3. Let $a, b \in R$ and $a^2 + b^2 \neq 0$. Suppose

$S = \left\{ z \in C : z = \frac{1}{a + ibt}, t \in R, t \neq 0 \right\}$, where $i = \sqrt{-1}$.

If $z = x + iy$ and $z \in S$, then (x, y) lies on

- (1) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0 \right)$ for $a > 0$, $b \neq 0$

- (2) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0 \right)$ for $a < 0$, $b \neq 0$

- (3) the x -axis for $a \neq 0$, $b = 0$

- (4) the y -axis for $a = 0$, $b \neq 0$ (JEE Advanced 2016)

4. Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies

$\operatorname{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is (are) possible value(s) of x ?

- (1) $-1 - \sqrt{1 - y^2}$ (2) $1 + \sqrt{1 + y^2}$
(3) $1 - \sqrt{1 + y^2}$ (4) $-1 + \sqrt{1 - y^2}$

(JEE Advanced 2017)

5. For a non-zero complex number z , let $\arg(z)$ denote the principal argument with $-\pi < \arg(z) \leq \pi$. Then, which of the following statement(s) is (are) FALSE?

- (1) $\arg(-1 - i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$
(2) The function $f: R \rightarrow (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in R$, is continuous at all points of R , where $i = \sqrt{-1}$
(3) For any two non-zero complex numbers z_1 and z_2 , $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π
(4) For any three given distinct complex numbers z_1, z_2 and z_3 , the locus of the point z satisfying the condition $\arg\left(\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}\right) = \pi$, lies on a straight line

(JEE Advanced 2018)

6. Let s, t, r be non-zero complex numbers and L be the set of solutions $z = x + iy$ ($x, y \in R, i = \sqrt{-1}$) of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE?

- (1) If L has exactly one element, then $|s| \neq |t|$
(2) If $|s| = |t|$, then L has infinitely many elements
(3) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2
(4) If L has more than one element, then L has infinitely many elements

(JEE Advanced 2018)

7. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE?

- (1) $\left|z + \frac{1}{2}\right| \leq \frac{1}{2}$ for all $z \in S$
(2) $|z| \leq 2$ for all $z \in S$
(3) $\left|z + \frac{1}{2}\right| \geq \frac{1}{2}$ for all $z \in S$
(4) The set S has exactly four elements

(JEE Advanced 2020)

8. For any complex number $w = c + id$, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such that for all

complex numbers $z = x + iy$ satisfying $\arg\left(\frac{z + \alpha}{z + \beta}\right) = \frac{\pi}{4}$,

the ordered pair (x, y) lies on the circle $x^2 + y^2 + 5x - 3y + 4 = 0$.

Then which of the following statements is (are) TRUE?

- (1) $\alpha = -1$ (2) $\alpha\beta = 4$
(3) $\alpha\beta = -4$ (4) $\beta = 4$

(JEE Advanced 2021)

Linked Comprehension Type

For Problems 1 and 2

Let $S = S_1 \cap S_2 \cap S_3$, where $S_1 = \{z \in C : |z| < 4\}$,

$$S_2 = \left\{z \in C : \operatorname{Im}\left[\frac{z - 1 + \sqrt{3}i}{1 - \sqrt{3}i}\right] > 0\right\} \text{ and } S_3 = \{z \in C : \operatorname{Re} z > 0\}$$

(JEE Advanced 2013)

1. Area of $A =$

- (1) $\frac{10\pi}{3}$ (2) $\frac{20\pi}{3}$
(3) $\frac{16\pi}{3}$ (4) $\frac{32\pi}{3}$

2. $\min_{z \in S} |1 - 3i - z| =$

- (1) $\frac{2 - \sqrt{3}}{2}$ (2) $\frac{2 + \sqrt{3}}{2}$
(3) $\frac{3 - \sqrt{3}}{2}$ (4) $\frac{3 + \sqrt{3}}{2}$

Matrix Match Type

1. Match the statements in List I with those in List II [Note: Here z takes the values in the complex plane and $\operatorname{Im}(z)$ and $\operatorname{Re}(z)$ denote, respectively, the imaginary part and the real part of z]

List I	List II
a. The set of points z satisfying $ z - i z - z + i z = 0$ is contained in or equal to	p. an ellipse with eccentricity $4/5$
b. The set of points z satisfying $ z + 4 + z - 4 = 10$ is contained in or equal to	q. the set of points z satisfying $\operatorname{Im} z = 0$
c. If $ \omega = 2$, then the set of points $z = \omega - (1/\omega)$ is contained in or equal to	r. the set of points z satisfying $ \operatorname{Im} z \leq 1$
d. If $ \omega = 1$, then the set of points $z = \omega + 1/\omega$ is contained in or equal to	s. the set of points z satisfying $ \operatorname{Re} z \leq 1$
	t. the set of points z satisfying $ z \leq 3$

(IIT-JEE, 2010)

2. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) - i\sin\left(\frac{2k\pi}{10}\right); k = 1, 2, \dots, 9$

List I	List II
a. For each z_k there exists a z_j such that $z_k \cdot z_j = 1$	p. True
b. There exists a $k \in \{1, 2, \dots, 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers	q. False
c. $\frac{ 1 - z_1 1 - z_2 \dots 1 - z_9 }{10}$ equals	r. 1
d. $1 - \sum_{k=0}^9 \cos\left(\frac{2k\pi}{10}\right)$ equals	s. 2

Codes:

- a b c d
 (1) p q r s
 (2) q s r p
 (3) s r p q
 (4) q s p r

(JEE Advanced 2014)

3. Match the statements/expressions given in List I with the values given in List II.

List I	List II
a. In R^2 , if the magnitude of the projection vector of the vector $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$, then possible value(s) of $ \alpha $ is (are)	p. 1
b. Let a and b be real numbers such that the function $f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$ is differentiable for all $x \in R$. Then possible value(s) of a is(are)	q. 2
c. Let $\omega \neq 1$ be a complex cube root of unity. If $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$, then possible value(s) of n is (are)	r. 3
d. Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that $a, 5, q, b$ is an arithmetic progression, then the value(s) of $ q - a $ is (are)	s. 4
	t. 5

(JEE Advanced 2015)

Numerical Value Type

1. Let ω be the complex number $\cos \frac{2\pi}{3} + i\sin \frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$
 is equal to _____.
 (IIT-JEE, 2010)

2. If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is _____.
 (IIT-JEE, 2011)

3. For any integer k , let $\alpha_k = \cos \frac{k\pi}{7} + i\sin \frac{k\pi}{7}$, where i

$$= \sqrt{-1}. \text{ Value of the expression } \frac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^3 |\alpha_{4k-1} - \alpha_{4k-2}|}$$
 is

_____.

(JEE Advanced 2015)

4. Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set $\{|a + b\omega + c\omega^2|^2; a, b, c \text{ are distinct non-zero integers}\}$ equals _____.

(JEE Advanced 2019)

5. For a complex number z , let $\text{Re}(z)$ denote the real part of z . Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with $\text{Re}(z_1) > 0$ and $\text{Re}(z_2) < 0$, is _____.

(JEE Advanced 2020)

Answers Key

EXERCISES

Single Correct Answer Type

1. (2)

6. (2)

11. (3)

16. (1)

21. (4)

26. (1)

31. (3)

36. (1)

41. (3)

46. (3)

51. (2)

56. (1)

61. (1)

66. (3)

71. (1)

76. (4)

81. (2)

86. (1)
2. (4)

7. (4)

12. (3)

17. (4)

22. (1)

27. (3)

32. (3)

37. (1)

42. (2)

47. (1)

52. (2)

57. (3)

62. (1)

67. (2)

72. (3)

77. (1)

82. (1)

87. (4)
3. (4)

8. (2)

13. (1)

18. (2)

23. (1)

28. (2)

33. (3)

38. (4)

43. (1)

48. (1)

53. (4)

58. (4)

63. (3)

68. (1)

73. (3)

78. (2)

83. (4)

88. (3)
4. (4)

9. (4)

14. (1)

19. (4)

24. (1)

29. (2)

34. (3)

39. (2)

44. (1)

49. (4)

54. (3)

59. (2)

64. (2)

69. (3)

74. (1)

79. (2)

84. (4)

89. (1)
5. (3)

10. (2)

15. (2)

20. (1)

25. (1)

30. (3)

35. (2)

40. (4)

45. (1)

50. (1)

55. (3)

60. (2)

65. (2)

70. (4)

75. (3)

80. (2)

85. (1)

Multiple Correct Answers Type

1. (1), (3)

3. (1), (2), (3), (4)

5. (1), (3), (4)

7. (1), (3)

9. (1), (3), (4)

11. (1), (2), (3)

13. (1), (4)

15. (1), (2), (3)

17. (1), (3)

19. (2), (3)

21. (1), (2), (3), (4)

23. (1), (2), (4)

25. (1), (2), (4)

27. (2), (3)

29. (1), (2), (4)

31. (1), (2)

33. (1), (2), (4)

35. (2), (3)

37. (1), (2)

39. (1), (2)

41. (1), (3)

43. (1), (2), (3), (4)

45. (1), (2), (3), (4)

47. (1), (2), (4)

49. (1), (2), (3)
2. (2), (3)

4. (1), (3)

6. (3), (4)

8. (3), (4)

10. (1), (2), (4)

12. (1), (3)

14. (1), (4)

16. (1), (4)

18. (2), (3)

20. (1), (4)

22. (1), (4)

24. (1), (2), (4)

26. (1), (2), (4)

28. (2), (3)

30. (1), (3), (4)

32. (1), (4)

34. (1), (4)

36. (1), (3), (4)

38. (1), (3)

40. (1), (2), (4)

42. (1), (2), (3), (4)

44. (1), (3), (4)

46. (1), (3), (4)

48. (1), (2), (3), (4)

Linked Comprehension Type

1. (1)

6. (2)

11. (3)

16. (4)

21. (4)

26. (1)

31. (3)
2. (4)

7. (3)

12. (2)

17. (4)

22. (2)

27. (2)

32. (4)
3. (3)

8. (3)

13. (3)

18. (4)

23. (3)

28. (2)

33. (3)
4. (4)

9. (1)

14. (1)

19. (3)

24. (2)

29. (1)

34. (3)
5. (2)

10. (2)

15. (4)

20. (3)

25. (4)

30. (2)

Matrix Match Type

1. $a \rightarrow q; b \rightarrow s; c \rightarrow p; d \rightarrow r;$

2. $a \rightarrow s; b \rightarrow p; c \rightarrow q; d \rightarrow r;$

3. $a \rightarrow q; b \rightarrow p; c \rightarrow s; d \rightarrow r;$

4. $a \rightarrow r; b \rightarrow p; c \rightarrow s; d \rightarrow q;$

5. (1)

6. (2)

Numerical Value Type

1. (3)

6. (3)

11. (5)

16. (18)

21. (15)

26. (100)
2. (21)

7. (4)

12. (6)

17. (30)

22. (0.5)

27. (12)
3. (1)

8. (17)

13. (3)

18. (448)

23. (1)

28. (2)
4. (15)

9. (1)

14. (20)

19. (5)

24. (40)

29. (2)
5. (2)

10. (1)

15. (125)

20. (33)

25. (32)

30. (1)

ARCHIVES

JEE ADVANCED

Single Correct Answer Type

1. (1)

2. (4)

3. (3)

4. (3)

5. (3)

Multiple Correct Answers Type

1. (1), (3), (4)

3. (1), (3), (4)

5. (1), (2), (4)

7. (2), (3)
2. (3), (4)

4. (1), (4)

6. (1), (3), (4)

8. (2), (4)

Linked Comprehension Type

1. (2)

2. (3)

Matrix Match Type

1. $a \rightarrow q; b \rightarrow p; c \rightarrow p, t; d \rightarrow q, t$

2. (1)

3. $c \rightarrow p, q, s, t$

Numerical Value Type

1. (1)

2. (5)

3. (4)

4. (3)

5. (8)

Chapter 3

Concept Application Exercises

Exercise 3.1

1. The said computation is not correct, because -2 and -3 both are negative and $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is true when at least one of a and b is positive or zero. The correct computation is

$$(\sqrt{-2})(\sqrt{-3}) = (i\sqrt{2})(i\sqrt{3}) = i^2\sqrt{6} = -\sqrt{6}$$

$$2. \frac{i^{584}(i^8 + i^6 + i^4 + i^2 + 1)}{i^{574}(i^8 + i^6 + i^4 + i^2 + 1)} - 1 = \frac{i^{584}}{i^{574}} - 1 = i^{10} - 1 = -1 - 1 = -2$$

3. Let $z = i^{[1+3+5+\dots+(2n+1)]}$.

Now, $1 + 3 + 5 + \dots + (2n + 1)$ are $n + 1$ terms of an A.P. whose sum is

$$\frac{n+1}{2}[1 + 2n + 1] = (n+1)^2$$

Hence, $z = i^{(n+1)^2}$. Now put $n = 1, 2, 3, 4, 5, \dots$

$$n = 1, z = i^4 = 1, n = 2, z = i^9 = i,$$

$$n = 3, z = i^{16} = 1, n = 4,$$

$$z = i^{25} = i, n = 5, z = i^{36} = 1, \dots$$

Thus, $z = \begin{cases} 1, & \text{if } n \text{ is odd} \\ i, & \text{if } n \text{ is even} \end{cases}$

4. We have,

$$x = -5 + 4i$$

$$\text{or } (x + 5)^2 = -16$$

$$\text{or } x^2 + 10x + 41 = 0$$

$$\text{Now, } x^4 + 9x^3 + 35x^2 - x + 4$$

$$= x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41)$$

$$+ 4(x^2 + 10x + 41) - 160$$

$$= 0x^2 - 0x + 4 \times 0 - 160 = -160 \quad [\text{Using (1)}]$$

Exercise 3.2

1. We have $z = 4 - 3i$

Multiplicative inverse of z is

$$\frac{1}{z} = \frac{1}{4 - 3i} = \frac{4 + 3i}{(4 - 3i)(4 + 3i)} = \frac{4 + 3i}{16 - 9i^2} = \frac{4 + 3i}{25} = \frac{4}{25} + \frac{3}{25}i$$

$$2. (a) \frac{(3 - 2i)(2 + 3i)}{(1 + 2i)(2 - i)} = \frac{(6 + 6) + i(-4 + 9)}{(2 + 2) + i(4 - 1)} = \frac{12 + 5i}{4 + 3i} = \frac{12 + 5i}{4 + 3i} \cdot \frac{4 - 3i}{4 - 3i} = \frac{(48 + 15) + i(-36 + 20)}{16 - 9i^2} = \frac{63}{25} - \frac{16}{25}i$$

$$(b) \frac{2 - \sqrt{-25}}{1 - \sqrt{-16}} = \frac{2 - 5i}{1 - 4i} = \frac{2 - 5i}{1 - 4i} \times \frac{1 + 4i}{1 + 4i} = \frac{(2 + 20) + i(8 - 5)}{1 - 16i^2} = \frac{22 + 3i}{17} = \frac{22}{17} + \frac{3}{17}i$$

$$3. \left(\frac{2i}{1+i}\right)^2 = \left(\frac{2i(1-i)}{(1+i)(1-i)}\right)^n = \left(\frac{2(i-i^2)}{2}\right)^n = (i+1)^n = (2i)^{n/2}$$

Hence, $n = 8$ is the least positive integer for which the given complex number is a positive integer.

$$4. z^2 - az + a - 1 = 0 \quad (1)$$

Putting $z = 1 + i$ in the equation, we get

$$a = 2 + i$$

$$\Rightarrow z^2 - (2 + i)z + 1 + i = 0 \text{ is the equation}$$

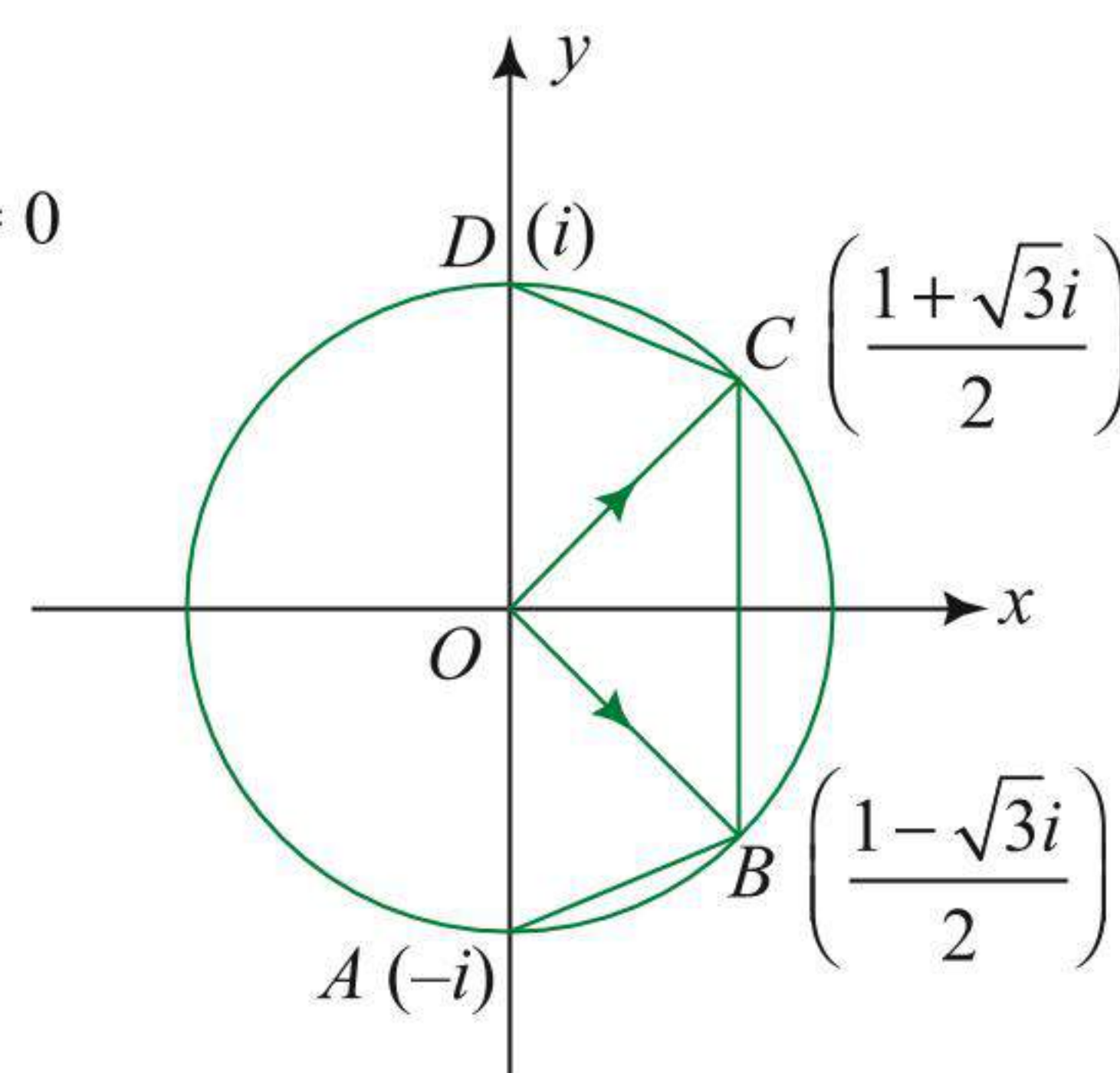
$$\Rightarrow z = 1 \text{ is the other root}$$

$$5. z^4 - z^3 + 2z^2 - z + 1 = 0$$

$$\Rightarrow (z^2 + 1)(z^2 - z + 1) = 0$$

$$\Rightarrow z = \pm i, \frac{1 \pm \sqrt{3}i}{2}$$

Plotting these points on the Argand plane, we find that $ABCD$ is an equilateral quadrilateral.



$$6. \text{ Let } Z = a + ib,$$

$$b \neq 0 \text{ where } \text{Im } Z = b$$

$$\therefore Z^5 = (a + ib)^5$$

$$= a^5 + 5a^4bi + 10a^3b^2i^2 + 10a^2b^3i^3 + 5ab^4i^4 + i^5b^5$$

$$\therefore \text{Im } Z^5 = 5a^4b - 10a^2b^3 + b^5$$

$$y = \frac{\text{Im } Z^5}{(\text{Im } Z)^5} = 5\left(\frac{a}{b}\right)^4 - 10\left(\frac{a}{b}\right)^2 + 1$$

$$= 5\left[\left(\frac{a}{b}\right)^4 - 2\left(\frac{a}{b}\right)^2 + \frac{1}{5}\right]$$

$$= 5\left[\left(\left(\frac{a}{b}\right)^2 - 1\right)^2 + \frac{1}{5} - 1\right]$$

$$= 5\left[\left(\frac{a}{b}\right)^2 - 1\right]^2 - 4$$

So, $y_{\min} = -4$

7. Let $z = (x - iy)(3 + 5i)$

$$= 3x + 5xi - 3yi - 5yi^2$$

$$= (3x + 5y) + i(5x - 3y)$$

$$\therefore \bar{z} = (3x + 5y) - i(5x - 3y)$$

It is given that, $\bar{z} = -6 - 24i$

$$\therefore (3x + 5y) - i(5x - 3y) = -6 - 24i$$

Equating real and imaginary parts, we get

$$3x + 5y = -6 \quad (1)$$

$$5x - 3y = 24 \quad (2)$$

Solving we get

$$x = 3, y = -3$$

8. $z_3 = (1 - \lambda)z_1 + z_2$

$$= \frac{(1 - \lambda)z_1 + \lambda z_2}{1 - \lambda + \lambda}$$

Hence, z_3 divides the line joining $A(z_1)$ and $B(z_2)$ in the ratio $\lambda: (1 - \lambda)$. Thus, the given points are collinear.

9. $(1 + i)^{n_1} + (1 + i^3)^{n_1} + (1 + i^5)^{n_2} + (1 + i^7)^{n_2}$

$$= (1 + i)^{n_1} + (1 - i)^{n_1} + (1 + i)^{n_2} + (1 - i)^{n_2}$$

$$= \left[(1 + i)^{n_1} + \overline{(1 + i)^{n_1}} \right] + \left[(1 + i)^{n_2} + \overline{(1 + i)^{n_2}} \right]$$

$$= \text{Purely real}$$

Exercise 3.3

1. We have,

$$(a + b) - i(3a + 2b) = 5 + 2i$$

$$\Rightarrow a + b = 5 \text{ and } -(3a + 2b) = 2$$

$$\Rightarrow a = -12, b = 17$$

2. Let $z = x + iy$. Then

$$\bar{z} = iz^2$$

$$\Rightarrow x - iy = i(x^2 - y^2 + 2ixy)$$

$$\Rightarrow x - iy = i(x^2 - y^2) - 2xy$$

$$\Rightarrow x(1 + 2y) = 0 \quad (1)$$

$$\text{and } x^2 - y^2 + y = 0 \quad (2)$$

From (1), $x = 0$ or $y = -1/2$. From (2), when $x = 0$, $y = 0, 1$ and when $y = -1/2$, $x = \pm(\sqrt{3}/2)$. For nonzero complex number z ,

$$z = i, \frac{\sqrt{3}}{2} - \frac{i}{2}, -\frac{\sqrt{3}}{2} - \frac{i}{2}$$

3. Let $i\alpha$ be a root where $\alpha \in R$

$$\Rightarrow -a\alpha^2 + bi\alpha + c + i = 0$$

$$\Rightarrow c = a\alpha^2, 1 + b\alpha = 0$$

$$\Rightarrow \alpha^2 = \frac{c}{a}, \alpha = \frac{-1}{b}$$

$$\Rightarrow \frac{1}{b^2} = \frac{c}{a} \Rightarrow a = b^2c$$

4. Let the roots be α and β .

$$\text{We have } \alpha + \beta = -(p + iq); \alpha\beta = 3i$$

$$\text{Given: } \alpha^2 + \beta^2 = 8$$

$$\text{or } (\alpha + \beta)^2 - 2\alpha\beta = 8$$

$$\text{or } (p + iq)^2 - 6i = 8$$

$$\text{or } p^2 - q^2 + i(2pq - 6) = 8$$

$$\text{or } p^2 - q^2 = 8 \text{ and } pq = 3$$

$$\Rightarrow p = 3 \text{ and } q = 1 \text{ or } p = -3 \text{ and } q = -1$$

5. Let $\sqrt{9 + 40i} = x + iy$. Then

$$(x + iy)^2 = 9 + 40i$$

$$\text{or } x^2 - y^2 = 9 \quad (1)$$

$$\text{and } xy = 20 \quad (2)$$

Squaring (1) and adding with 4 times the square of (2), we get

$$x^4 + y^4 - 2x^2y^2 + 4x^2y^2 = 81 + 1600$$

$$\text{or } (x^2 + y^2)^2 = 1681$$

$$\text{or } x^2 + y^2 = 41 \quad (3)$$

From (1) + (3), we get

$$x^2 = 25 \text{ or } x = \pm 5$$

$$\text{and } y = \pm 4$$

From Eq. (2), we can see that x and y are of same sign. Therefore,

$$x + iy = (5 + 4i) \text{ or } -(5 + 4i)$$

6.
$$\frac{(\sqrt{5+12i} + \sqrt{5-12i})(\sqrt{5+12i} + \sqrt{5-12i})}{(\sqrt{5+12i} - \sqrt{5-12i})(\sqrt{5+12i} + \sqrt{5-12i})}$$

$$= \frac{5 + 12i + 5 - 12i + 2\sqrt{5+12i}\sqrt{5-12i}}{5 + 12i - 5 + 12i}$$

$$= \frac{10 + 2 \times 13}{24i}$$

$$= -\frac{3}{2}i$$

7. $\sqrt{x + iy} = \pm(a + bi)$

$$\text{or } x + iy = a^2 - b^2 + 2iab$$

$$\text{or } x = a^2 - b^2, y = 2ab$$

$$\therefore \sqrt{-x - iy} = \sqrt{-(a^2 - b^2) - 2iab}$$

$$= \sqrt{b^2 + (ia)^2 - 2iab}$$

$$= \sqrt{(b - ia)^2} = \pm(b - ia)$$

Exercise 3.4

1. Complex cube roots of unity are ω, ω^2 . Let $\alpha = \omega, \beta = \omega^2$. Then

$$\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = \omega^4 + (\omega^2)^4 + (\omega^{-1})(\omega^2)^{-1}$$

$$= \omega + \omega^2 + 1 = 0$$

2. $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \dots$ up to $2n$ factors

$$= (-\omega^2)(-\omega)(1 + \omega)(1 + \omega^2) \dots \text{ up to } 2n \text{ factors}$$

$$= 1 \times 1 \times 1 \times \dots \text{ up to } n \text{ factors} = 1$$

3. (a) Here, $-1/2 + (1/2)i\sqrt{3}$ is one of the two imaginary cube roots of unity. If we denote it by ω , then

$$\omega^{1000} = \omega^{999} \omega = (\omega^3)^{333} \omega = \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

(b)
$$z = \frac{(\sqrt{3} + i)^{17}}{(1 - i)^{50}}$$

$$= \frac{1}{i^{17}} \frac{(i\sqrt{3} + i^2)^{17}}{[(1 - i)^2]^{25}}$$

$$= \frac{2^{17}}{i} \left(\frac{i\sqrt{3} - 1}{2} \right)^{17}$$

$$= \frac{1}{2^8} (\omega)^{17}$$

$$= \frac{1}{2^8} (\omega)^2$$

$$= \frac{1}{2^8} \left(\frac{-1 - i\sqrt{3}}{2} \right)$$

$$\begin{aligned}
 \text{(c)} \quad & (i + \sqrt{3})^{100} + (i - \sqrt{3})^{100} + 2^{100} \\
 &= \left(\frac{i^2 + i\sqrt{3}}{i} \right)^{100} + \left(\frac{i^2 - i\sqrt{3}}{i} \right)^{100} + 2^{100} \\
 &= \frac{2^{100}}{i^{100}} \left(\frac{-1 + i\sqrt{3}}{2} \right)^{100} + \frac{2^{100}}{i^{100}} \left(\frac{-1 - i\sqrt{3}}{2} \right)^{100} + 2^{100} \\
 &= 2^{100}(\omega)^{100} + 2^{100}(\omega^2)^{100} + 2^{100} \\
 &= 2^{100}(\omega^{100} + \omega^{200} + 1) \\
 &= 2^{100}(\omega + \omega^2 + 1) = 0
 \end{aligned}$$

$$4. \quad z + z^{-1} = 1$$

$$\text{or} \quad z^2 - z + 1 = 0$$

$$\Rightarrow \quad z = -\omega \text{ or } -\omega^2$$

$$\text{For } z = -\omega,$$

$$z^{100} + z^{-100} = (-\omega)^{100} + (-\omega)^{-100}$$

$$= \omega + \frac{1}{\omega} = \omega + \omega^2 = -1$$

$$\text{For } z = -\omega^2,$$

$$z^{100} + z^{-100} = (-\omega^2)^{100} + (-\omega^2)^{-100}$$

$$= \omega^{200} + \frac{1}{\omega^{200}}$$

$$= \omega^2 + \frac{1}{\omega^2} = \omega^2 + \omega = -1$$

$$5. \quad x^{12} - 1 = (x^6 + 1)(x^6 - 1) = (x^6 + 1)(x^2 - 1)(x^4 + x^2 + 1)$$

Common roots are given by $x^4 + x^2 + 1 = 0$

$$\therefore \quad x^2 = \frac{-1 \pm i\sqrt{3}}{2} = \omega, \omega^2 \text{ or } \omega^4, \omega^2 \quad (\because \omega^3 = 1)$$

$$\text{or} \quad x = \pm \omega^2, \pm \omega$$

$$6. \quad x^3 - 3x^2 + 3x + 7 = 0$$

$$\Rightarrow \quad (x - 1)^3 = -8$$

$$\Rightarrow \quad \frac{x - 1}{-2} = (1)^{1/3} = 1, \omega, \omega^2$$

$$\Rightarrow \quad \alpha = -1, \beta = 1 - 2\omega, \gamma = 1 - 2\omega^2$$

$$\begin{aligned}
 \therefore \quad E &= \frac{-2}{-2\omega} + \frac{-2\omega}{-2\omega^2} + \frac{-2\omega^2}{-2} \\
 &= \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} = \frac{3}{\omega} = 3\omega^2
 \end{aligned}$$

$$7. \quad \text{Put } t + 1 = Z.$$

Then $t^2 + 3t + 3 = (t + 1)^2 + (t + 2) = Z^2 + Z + 1 = (Z - \omega)(Z - \omega^2)$, where ω is a complex cube root of unity.

When $Z = \omega$, the expression $(t + 1)^{n+1} + (t + 2)^{2n-1}$ becomes

$$\begin{aligned}
 \omega^{n+1} + (\omega + 1)^{2n-1} &= \omega^{n+1} + (-\omega^2)^{2n-1} \\
 &= \omega^{n+1} + (-1)^{2n-1} \omega^{4n-2} \\
 &= \omega^{n+1} \{1 + (-1)^{2n-1} \omega^{3n-3}\} \\
 &= \omega^{n+1} (1 - 1) = 0 \text{ as } \omega^{3n-3} = 1
 \end{aligned}$$

$\therefore \quad Z - \omega$ is a factor of the expression.

Similarly, $Z - \omega^2$ is also a factor of the expression,

$$\begin{aligned}
 \text{(b)} \quad z &= \frac{1 + \sqrt{3}i}{\sqrt{3} + i} = \frac{(1 + \sqrt{3}i)(\sqrt{3} - i)}{(\sqrt{3} + i)(\sqrt{3} - i)} \\
 &= \frac{2\sqrt{3} + 2i}{4} = \frac{\sqrt{3} + i}{2}
 \end{aligned}$$

$$\arg(z) = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

$$\text{(c)} \quad z = \sin \alpha + i(1 - \cos \alpha)$$

$$\begin{aligned}
 \Rightarrow \quad \arg(z) &= \tan^{-1} \left(\frac{1 - \cos \alpha}{\sin \alpha} \right) = \tan^{-1} \left(\frac{2 \sin^2 \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} \right) \\
 &= \tan^{-1} \tan \left(\frac{\alpha}{2} \right) = \frac{\alpha}{2}
 \end{aligned}$$

$$\text{(d)} \quad z = (1 + i\sqrt{3})^2 = 1 - 3 + 2i\sqrt{3} = -2 + i(2\sqrt{3})$$

So, z lies in second quadrant.

$$\begin{aligned}
 \therefore \quad \arg(z) &= \pi - \tan^{-1} \left| \frac{2\sqrt{3}}{-2} \right| \\
 &= \pi - \tan^{-1} \sqrt{3} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}
 \end{aligned}$$

$$2. \quad z = (\tan 1 - i)^2 = (\tan^2 1 - 1) - (2 \tan 1)i$$

$$|z| = \sqrt{(\tan^2 1 - 1)^2 + 4 \tan^2 1} = \sqrt{(\tan^2 1 + 1)^2} = \sec^2 1$$

Since $\tan^2 1 - 1 < 0$ and $-2 \tan 1 < 0$, so, z lies in the third quadrant.

$$\Rightarrow \quad \arg(z) = -\pi + \tan^{-1} \left| \frac{2 \tan 1}{1 - \tan^2 1} \right| = -\pi + \tan^{-1} |\tan 2| = 2 - \pi$$

$$3. \quad z = (1 - \cos 2\alpha) + i \sin 2\alpha$$

$$= 2 \sin^2 \alpha + i(2 \sin \alpha \cos \alpha)$$

$$= 2 \sin \alpha (\sin \alpha + i \cos \alpha)$$

$$= -2 \sin \alpha (-\sin \alpha - i \cos \alpha)$$

$$= -2 \sin \alpha \left(\cos \left(\frac{3\pi}{2} - \alpha \right) + i \sin \left(\frac{3\pi}{2} - \alpha \right) \right)$$

Thus, $|z| = -2 \sin \alpha$.

$$\text{Also, } \frac{3\pi}{2} - \alpha \in \left(\frac{-\pi}{2}, 0 \right)$$

Therefore, z lies in the fourth quadrant.

$$\therefore \quad \arg(z) = \frac{3\pi}{2} - \alpha$$

$$4. \quad Z = \sin \frac{6\pi}{5} + i \left(1 + \cos \frac{6\pi}{5} \right)$$

$$= 2 \cos \frac{3\pi}{5} \left(\sin \frac{3\pi}{5} + i \cos \frac{3\pi}{5} \right)$$

$$= 2 \cos \frac{3\pi}{5} \left(\cos \left(\frac{\pi}{2} - \frac{3\pi}{5} \right) + i \sin \left(\frac{\pi}{2} - \frac{3\pi}{5} \right) \right)$$

$$= 2 \cos \frac{3\pi}{5} \left[\cos \left(\frac{-\pi}{10} \right) + i \sin \left(\frac{-\pi}{10} \right) \right]$$

$$= -2 \cos \frac{3\pi}{5} \left[-\cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right]$$

$$= -2 \cos \frac{3\pi}{5} \left[\cos \left(\pi - \frac{\pi}{10} \right) + i \sin \left(\pi - \frac{\pi}{10} \right) \right]$$

Exercise 3.5

$$1. \quad \text{(a)} \quad \text{Let } z = -1 - i\sqrt{3}. \text{ Then } \alpha = \tan^{-1} \left| \frac{b}{a} \right| = \tan^{-1} \left| \frac{\sqrt{3}}{1} \right| = \frac{\pi}{3}.$$

Clearly, z lies in III quadrant.

$$\text{Therefore, argument } \theta = -(\pi - \alpha) = -(\pi - \pi/3) = \frac{-2\pi}{3}$$

$$= -2 \cos \frac{3\pi}{5} \left[\cos \frac{9\pi}{10} + i \sin \frac{9\pi}{10} \right]$$

$$\therefore \arg(Z) = \frac{9\pi}{10}; \text{ and } |z| = -2 \cos \frac{3\pi}{5}$$

$$5. z = re^{i\theta} = r(\cos \theta + i \sin \theta)$$

$$\Rightarrow iz = ir(\cos \theta + i \sin \theta)$$

$$= -r \sin \theta + ir \cos \theta$$

$$\Rightarrow e^{iz} = e^{(-r \sin \theta + ir \cos \theta)}$$

$$= e^{-r \sin \theta} e^{ri \cos \theta}$$

$$\Rightarrow |e^{iz}| = |e^{-r \sin \theta}| |e^{ri \cos \theta}|$$

$$= e^{-r \sin \theta} |e^{i\alpha}|,$$

$$= e^{-r \sin \theta} [\cos^2 \alpha + \sin^2 \alpha]^{1/2}$$

$$= e^{-r \sin \theta}$$

$$\text{where } \alpha = r \cos \theta$$

$$6. z = x + iy$$

$$\text{or } z^2 = x^2 - y^2 + 2ixy$$

$$\Rightarrow \operatorname{Re}(z^2) = x^2 - y^2$$

$$\text{Also, } |z| = \sqrt{x^2 + y^2}$$

$$\Rightarrow x^2 - y^2 = 0, x^2 + y^2 = 3$$

$$\Rightarrow x^2 = y^2 = \frac{3}{2}$$

$$\Rightarrow x = \pm \sqrt{\frac{3}{2}}, y = \pm \sqrt{\frac{3}{2}}$$

$$\Rightarrow z = \pm \sqrt{\frac{3}{2}} \pm \sqrt{\frac{3}{2}} i$$

Thus, there are four complex numbers.

$$7. z = x + iy$$

$$\Rightarrow \operatorname{Re}(z) = x, \operatorname{Im}(z) = y$$

$$|z - i\operatorname{Re}(z)| = |z - \operatorname{Im}(z)|$$

$$\Rightarrow |x + iy - ix| = |x + iy - y|$$

$$\Rightarrow x^2 + (x - y)^2 = (x - y)^2 + y^2$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow |x| = |y|$$

Hence, z lies on the bisectors the quadrants.

$$8. |z + 5|^2 - |z - 5|^2 = 10$$

$$\text{or } (z + 5)(\bar{z} + 5) - (z - 5)(\bar{z} - 5) = 10$$

$$\text{or } 5(z + \bar{z}) + 25 + 5(z + \bar{z}) - 25 = 10$$

$$\text{or } 2 \times 2x = 2$$

$$\text{or } x = \frac{1}{2}$$

which is the equation of a straight line.

$$9. z^2 + |z| = 0$$

$$\Rightarrow x^2 - y^2 + i(2xy) + \sqrt{x^2 + y^2} = 0$$

$$\Rightarrow x^2 - y^2 + \sqrt{x^2 + y^2} = 0$$

$$\text{and } 2xy = 0$$

From (2), let $x = 0$. From (1),

$$-y^2 + \sqrt{y^2} = 0$$

$$\Rightarrow -|y|^2 + |y| = 0$$

$$\Rightarrow |y| = 0 \text{ or } 1$$

$$\Rightarrow y = 0 \text{ or } y = \pm 1$$

From (2), if $y = 0$, then from (1),

$$x^2 + \sqrt{x^2} = 0$$

$$\Rightarrow |x|^2 + |x| = 0$$

$$\Rightarrow x = 0$$

Hence, complex numbers are $0 + i0, 0 + i, 0 - i$.

$$10. z = x + iy$$

$$A = \{z : |z| \leq 2\}$$

$$\Rightarrow \sqrt{x^2 + y^2} \leq 2$$

$$\text{or } x^2 + y^2 \leq 4$$

$$\Rightarrow z \text{ lies on or inside the circle } x^2 + y^2 = 4$$

$$B = \{z : (1 - i)z + (1 + i)\bar{z} \geq 4\}$$

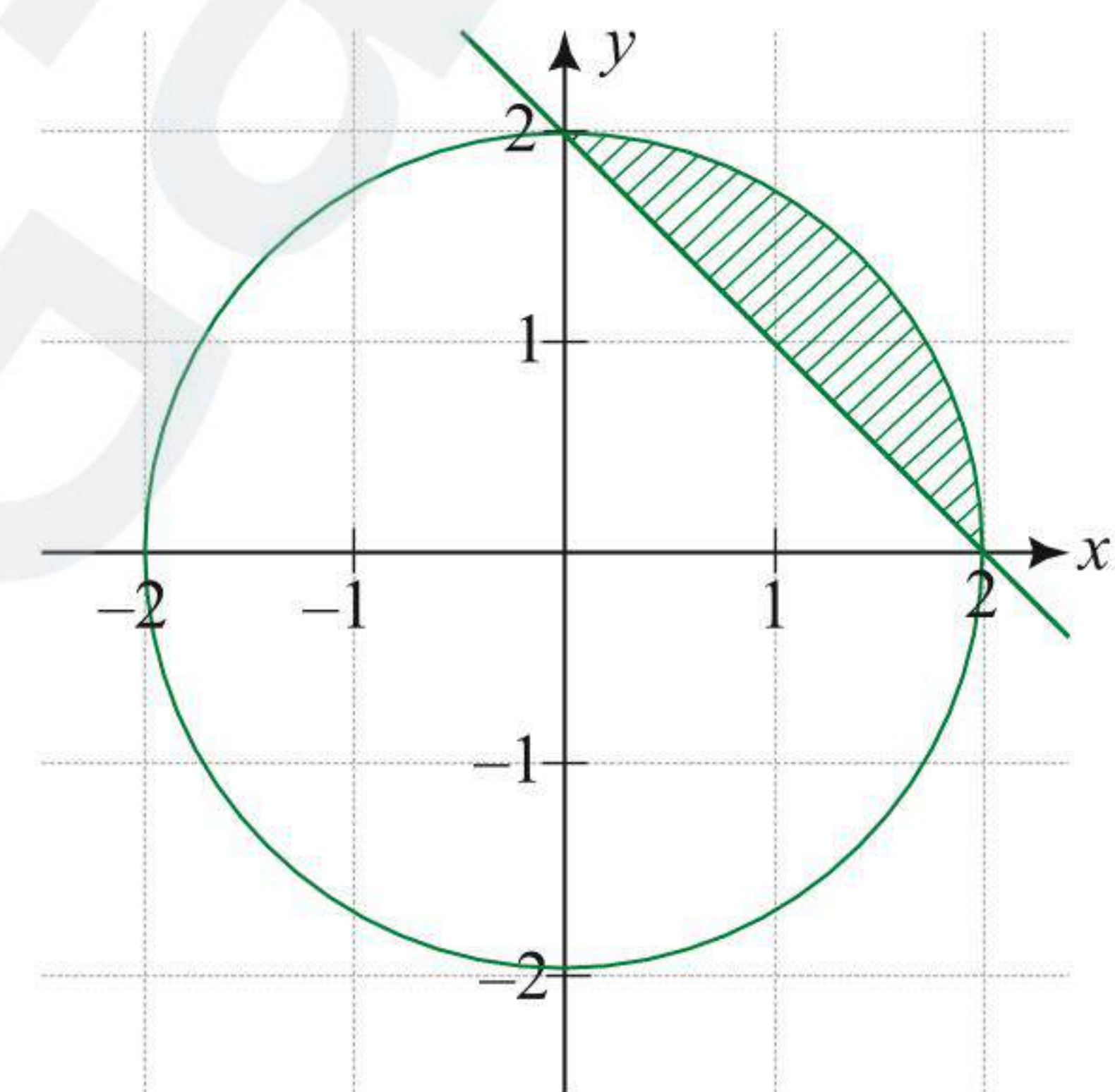
$$\Rightarrow (1 - i)(x + iy) + (1 + i)(x - iy) \geq 4$$

$$\Rightarrow x + iy - ix + y + x - iy + ix + y \geq 4$$

$$\Rightarrow x + y \geq 2$$

Area of region $A \cap B$ is the shaded region shown in the figure.

$$\text{Area} = \frac{\pi(2)^2}{4} - \frac{1}{2} \times 2 \times 2 = \pi - 2$$



$$11. e^{e^{i\theta}} = e^{\cos \theta + i \sin \theta} = e^{\cos \theta} [e^{i \sin \theta}]$$

$$= e^{\cos \theta} [\cos(\sin \theta) + i \sin(\sin \theta)]$$

Therefore, the real part is $e^{\cos \theta} [\cos(\sin \theta)]$.

$$12. z = i^i = \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^i = (e^{i\pi/2})^i = e^{-\frac{\pi}{2}}$$

$$\Rightarrow \operatorname{Re}(z) = e^{-\frac{\pi}{2}}$$

Exercise 3.6

$$1. |z_1| = \left| \frac{1-i}{1+i\sqrt{3}} \right|^{\frac{1}{6}} = \left| \frac{\sqrt{2}}{2} \right|^{\frac{1}{6}} = 2^{-\frac{1}{12}}$$

Similarly,

$$|z_2| = 2^{-\frac{1}{12}}; |z_3| = 2^{-\frac{1}{12}}$$

Hence, the result.

2. We have,

$$\sqrt{3} + i = (a + ib)(c + id)$$

$$\Rightarrow ac - bd = \sqrt{3} \text{ and } ad + bc = 1$$

$$\text{Now, } \tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{d}{c}\right) = \tan^{-1}\left(\frac{\frac{b}{a} + \frac{d}{c}}{1 - \frac{b}{a} \frac{d}{c}}\right)$$

$$= \tan^{-1}\left(\frac{bc + ad}{ac - bd}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = n\pi + \frac{\pi}{6}, n \in \mathbb{Z}$$

3. We have $z_2 = \bar{z}_1$ and $z_4 = \bar{z}_3$. Therefore,

$$z_1 z_2 = |z_1|^2 \text{ and } z_3 z_4 = |z_3|^2$$

$$\begin{aligned} \text{Now, } \arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) &= \arg\left(\frac{z_1 z_2}{z_4 z_3}\right) \\ &= \arg\left(\frac{|z_1|^2}{|z_3|^2}\right) \\ &= \arg\left(\left|\frac{z_1}{z_3}\right|^2\right) = 0 \end{aligned}$$

$$\begin{aligned} 4. \quad z &= \frac{i-1}{i\left(1 - \cos\frac{2\pi}{5}\right) + \sin\frac{2\pi}{5}} \\ &= \frac{i-1}{i2\sin^2\frac{\pi}{5} + 2\sin\frac{\pi}{5}\cos\frac{\pi}{5}} \\ &= \frac{i-1}{\left(2\sin\frac{\pi}{5}\right)\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)} \\ |z| &= \frac{|i-1|}{\left(2\sin\frac{\pi}{5}\right)\left|\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right|} \\ &= \frac{\sqrt{2}}{\left(2\sin\frac{\pi}{5}\right)\left|\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right|} \\ &= \frac{1}{\sqrt{2}} \operatorname{cosec}\frac{\pi}{5} \end{aligned}$$

$$\begin{aligned} \arg z &= \arg\left[\frac{i-1}{\left(2\sin\frac{\pi}{5}\right)\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)}\right] \\ &= \arg(-1+i) - \arg\left(2\sin\frac{\pi}{5}\right) - \arg\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right) \\ &= \frac{3\pi}{4} - 0 - \frac{\pi}{5} = \frac{11\pi}{20} \end{aligned}$$

5. We have,

$$\begin{aligned} (1+i)(1+2i)(1+3i)\cdots(1+ni) &= x+iy \\ \Rightarrow |(1+i)(1+2i)\cdots(1+ni)| &= |x+iy| \\ \Rightarrow |1+i| |1+2i| \cdots |1+ni| &= |x+iy| \\ &[\because |z_1 z_2 \cdots z_n| = |z_1| |z_2| \cdots |z_n|] \\ \Rightarrow \sqrt{1+1}\sqrt{1+4}\cdots\sqrt{1+n^2} &= \sqrt{x^2+y^2} \\ \Rightarrow 2 \times 5 \times 10 \cdots (1+n^2) &= (x^2+y^2) \quad [\text{On squaring both sides}] \end{aligned}$$

$$6. \quad a+ib = \frac{(x+i)^2}{2x^2+1}$$

$$\text{or } |a+ib| = \left|\frac{(x+i)^2}{2x^2+1}\right|$$

$$\text{or } \sqrt{a^2+b^2} = \frac{|x+i|^2}{2x^2+1}$$

$$\text{or } a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$$

$$7. \quad (z^3+3)^2 = 16i^2$$

$$z^3+3 = 4i \text{ or } -4i$$

$$z^3 = -3+4i \text{ or } -3-4i$$

$$|z^3| = |-3+4i| = 5$$

$$|z|^3 = 5$$

$$\Rightarrow |z| = 5^{1/3}$$

$$8. \quad z_1^2 + z_2^2 + 2z_1 z_2 \cos \theta = 0$$

$$\text{or } \left(\frac{z_1}{z_2}\right)^2 + 2\left(\frac{z_1}{z_2}\right) \cos \theta + 1 = 0$$

$$\text{or } \left(\frac{z_1}{z_2} + \cos \theta\right)^2 = -(1 - \cos^2 \theta) = -\sin^2 \theta$$

$$\text{or } \frac{z_1}{z_2} = -\cos \theta \pm i \sin \theta$$

$$\text{or } \left|\frac{z_1}{z_2}\right| = \sqrt{(-\cos \theta)^2 + \sin^2 \theta} = 1$$

$$\text{or } |z_1| = |z_2|$$

$$\text{or } |z_1 - 0| = |z_2 - 0|$$

Thus, triangle with vertices O, z_1, z_2 is isosceles.

$$9. \quad \text{Here, } \left|\frac{z_2 - z_0}{z_0 - z_1}\right| = 1 \quad \text{and} \quad \arg\left(\frac{z_2 - z_0}{z_0 - z_1}\right) = \frac{\pi}{2}$$

$$\therefore \frac{z_2 - z_0}{z_0 - z_1} = 1 \left\{ \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right\} = i$$

$$\Rightarrow z_2 - z_0 = iz_0 - iz_1$$

$$\Rightarrow z_2 + iz_1 = (i+1)z_0$$

$$\begin{aligned} \text{or } z_0 &= \frac{z_2 + iz_1}{1+i} \\ &= \frac{(z_2 + iz_1)(1-i)}{1^2 + 1^2} \end{aligned}$$

$$= \frac{1}{2} \{(i+1)z_1 + (1-i)z_2\}$$

$$10. \quad \left|\frac{z-2}{z-3}\right| = 2$$

$$\Rightarrow |z-2|^2 = 4|z-3|^2$$

$$\Rightarrow |x-2+iy|^2 = 4|x-3+iy|^2$$

$$\Rightarrow (x-2)^2 + y^2 = 4[(x-3)^2 + y^2]$$

$$\Rightarrow 3x^2 + 3y^2 - 24x + 4x + 36 - 4 = 0$$

$$\Rightarrow x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

$$\Rightarrow \left(x - \frac{10}{3}\right)^2 + y^2 = \frac{4}{9}$$

Thus, distance of (x, y) from the point $(10/3, 0)$ is $2/3$.

So, centre of the circle is $(10/3, 0)$ and radius is $2/3$.

Exercise 3.7

$$\begin{aligned} 1. \quad (a) \quad \frac{(\cos \alpha + i \sin \alpha)^4}{(\sin \beta + i \cos \beta)^5} &= \frac{\cos 4\alpha + i \sin 4\alpha}{i^5 (\cos \beta - i \sin \beta)^5} \\ &= -i(\cos 4\alpha + i \sin 4\alpha)(\cos \beta - i \sin \beta)^{-5} \\ &= -i[\cos 4\alpha + i \sin 4\alpha][\cos 5\beta + i \sin 5\beta] \\ &= -i[\cos(4\alpha + 5\beta) + i \sin(4\alpha + 5\beta)] \\ &= \sin(4\alpha + 5\beta) - i \cos(4\alpha + 5\beta) \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \left(\frac{1 + \cos \phi + i \sin \phi}{1 + \cos \phi - i \sin \phi} \right)^n \\
 &= \left[\frac{2 \cos^2(\phi/2) + i \sin(\phi/2) \cos(\phi/2)}{2 \cos^2(\phi/2) - 2i \sin(\phi/2) \cos(\phi/2)} \right]^n \\
 &= \left[\frac{\cos(\phi/2) + i \sin(\phi/2)}{\cos(\phi/2) - i \sin(\phi/2)} \right]^n \\
 &= \left[\frac{(\cos \phi + i \sin \phi)^{1/2}}{(\cos \phi + i \sin \phi)^{-1/2}} \right]^n \\
 &= [\cos \phi + i \sin \phi]^n \\
 &= \cos n\phi + i \sin n\phi
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \frac{(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta)}{(\cos \gamma + i \sin \gamma)(\cos \delta + i \sin \delta)} \\
 &= \cos(\alpha + \beta - \gamma - \delta) + i \sin(\alpha + \beta - \gamma - \delta)
 \end{aligned}$$

$$2. \text{ Let } \cos \frac{\pi}{10} - i \sin \frac{\pi}{10} = z$$

$$\Rightarrow \cos \frac{\pi}{10} + i \sin \frac{\pi}{10} = \frac{1}{z}$$

$$\begin{aligned}
 \Rightarrow \quad & \left[\frac{1 - \cos \frac{\pi}{10} + i \sin \frac{\pi}{10}}{1 - \cos \frac{\pi}{10} - i \sin \frac{\pi}{10}} \right]^{10} = \left(\frac{1 - z}{1 - \frac{1}{z}} \right)^{10} \\
 &= \left\{ \frac{-(z-1)z}{(z-1)} \right\}^{10} \\
 &= (-z)^{10} \\
 &= z^{10} \\
 &= \left(\cos \frac{\pi}{10} - i \sin \frac{\pi}{10} \right)^{10} \\
 &= \cos \pi - i \sin \pi \\
 &= -1
 \end{aligned}$$

$$3. \quad iz^4 = -1$$

$$z^4 = \frac{-1}{i}$$

$$\text{or } z^4 = i$$

$$\text{or } z = (i)^{1/4}$$

$$\text{or } z = (0 + i)^{1/4}$$

$$\text{or } z = \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^{1/4} = \cos \frac{\pi}{8} + i \sin \frac{\pi}{8}$$

$$4. \quad (1+i)^n + (1-i)^n$$

$$= (2)^{n/2} \left\{ \cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} + \cos \frac{n\pi}{4} - i \sin \frac{n\pi}{4} \right\}$$

$$= 2^{n/2} 2 \cos \frac{n\pi}{4} = 2^{n/2+1} \cos \frac{n\pi}{4} = (\sqrt{2})^{n+2} \cos \frac{n\pi}{4}$$

$$5. \text{ Let } a = r \cos \theta, b = r \sin \theta$$

$$\text{Then } z = (a + ib)^5 + (b + ia)^5$$

$$= r^5 \{ (\cos \theta + i \sin \theta)^5 + (\sin \theta + i \cos \theta)^5 \}$$

$$= r^5 \left[(\cos 5\theta + i \sin 5\theta) + \left\{ \cos \left(\frac{\pi}{2} - \theta \right) + i \sin \left(\frac{\pi}{2} - \theta \right) \right\}^5 \right]$$

$$= r^5 \left[(\cos 5\theta + i \sin 5\theta) + \cos 5 \left(\frac{\pi}{2} - \theta \right) + i \sin 5 \left(\frac{\pi}{2} - \theta \right) \right]$$

$$= r^5 [(\cos 5\theta + \sin 5\theta)(1 + i)]$$

$$\text{Clearly } \operatorname{Re}(z) = \operatorname{Im}(z) = r^5 (\cos 5\theta + \sin 5\theta)$$

$$6. \text{ Let } z_1 = \cos \alpha + i \sin \alpha, z_2 = \cos \beta + i \sin \beta,$$

$$z_3 = \cos \gamma + i \sin \gamma$$

$$\therefore z_1 + z_2 + z_3 = (\cos \alpha + \cos \beta + \cos \gamma) + i(\sin \alpha + \sin \beta + \sin \gamma) = 0 + i \times 0 = 0 \quad (1)$$

$$\text{(a) Now, } \frac{1}{z_1} = (\cos \alpha + i \sin \alpha)^{-1} = \cos \alpha - i \sin \alpha$$

$$\frac{1}{z_2} = \cos \beta - i \sin \beta$$

$$\frac{1}{z_3} = \cos \gamma - i \sin \gamma$$

$$\therefore \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}$$

$$= (\cos \alpha + \cos \beta + \cos \gamma) - i(\sin \alpha + \sin \beta + \sin \gamma) = 0 - i \times 0 = 0 \quad (2)$$

$$z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1 z_2 + z_2 z_3 + z_3 z_1)$$

$$= 0 - 2z_1 z_2 z_3 \left(\frac{1}{z_3} + \frac{1}{z_1} + \frac{1}{z_2} \right)$$

$$= 0 - 2z_1 z_2 z_3 \times 0 = 0 \quad [\text{Using (1) and (2)}]$$

$$\Rightarrow (\cos \alpha + i \sin \alpha)^2 + (\cos \beta + i \sin \beta)^2 + (\cos \gamma + i \sin \gamma)^2 = 0$$

$$\Rightarrow (\cos 2\alpha + i \sin 2\alpha) + (\cos 2\beta + i \sin 2\beta) + (\cos 2\gamma + i \sin 2\gamma) = 0 + i \times 0$$

Equating real and imaginary parts on both sides,

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$$

$$\text{and } \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$$

$$\text{(b) } z_1^3 + z_2^3 + z_3^3 = (z_1 + z_2)^3 - 3z_1 z_2 (z_1 + z_2) + z_3^3 = (-z_3)^3 - 3z_1 z_2 (-z_3) + z_3^3 \quad [\text{Using (1)}]$$

$$= 3z_1 z_2 z_3$$

$$\Rightarrow (\cos \alpha + i \sin \alpha)^3 + (\cos \beta + i \sin \beta)^3 + (\cos \gamma + i \sin \gamma)^3$$

$$= 3(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta)(\cos \gamma + i \sin \gamma)$$

$$\Rightarrow \cos 3\alpha + i \sin 3\alpha + \cos 3\beta + i \sin 3\beta + \cos 3\gamma + i \sin 3\gamma$$

$$= 3\{\cos(\alpha + \beta + \gamma) + i \sin(\alpha + \beta + \gamma)\}$$

Equating imaginary parts on both sides,

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin(\alpha + \beta + \gamma)$$

(c) Equating real parts on both sides,

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos(\alpha + \beta + \gamma)$$

Exercise 3.8

$$1. \quad a, b, c \text{ are complex numbers on the unit circle } |z| = 1$$

$$\therefore |a| = |b| = |c| = 1$$

$$\therefore a\bar{a} = b\bar{b} = c\bar{c} = 1$$

$$abc = a + b + c$$

$$\Rightarrow |abc| = |a + b + c|$$

$$= |\bar{a} + \bar{b} + \bar{c}|$$

$$= \left| \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right|$$

$$= \left| \frac{ab + bc + ca}{abc} \right|$$

$$\therefore |ab + bc + ca| = |abc| |abc| = (|a| |b| |c|)^2 = 1$$

$$2. \frac{1+z+z^2}{1-z+z^2} = \frac{1-z+z^2+2z}{1-z+z^2}$$

$$= 1 + \frac{2}{z-1+(1/z)} \text{ is real}$$

$$\text{Hence, } z + \frac{1}{z} \in \mathbb{R}$$

$$\Rightarrow z + \frac{1}{z} = \bar{z} + \frac{1}{\bar{z}}$$

$$\Rightarrow (z - \bar{z}) = \frac{1}{\bar{z}} - \frac{1}{z} = \frac{z - \bar{z}}{|z|^2}$$

$$\Rightarrow (z - \bar{z})(|z|^2 - 1) = 0$$

$$\Rightarrow |z| = 1 \text{ (} z = \bar{z} \text{ is not possible as } z \text{ is not real)}$$

$$3. \frac{a}{|z_1 - z_2|} = \frac{b}{|z_2 - z_3|} = \frac{c}{|z_3 - z_1|} = \lambda \text{ (say)}$$

$$\Rightarrow a^2 = \lambda^2(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$b^2 = \lambda^2(z_2 - z_3)(\bar{z}_2 - \bar{z}_3)$$

$$c^2 = \lambda^2(z_3 - z_1)(\bar{z}_3 - \bar{z}_1)$$

$$\Rightarrow \frac{a^2}{z_1 - z_2} + \frac{b^2}{z_2 - z_3} + \frac{c^2}{z_3 - z_1} = \lambda^2(\bar{z}_1 - \bar{z}_2 + \bar{z}_2 - \bar{z}_3 + \bar{z}_3 - \bar{z}_1) = 0$$

$$4. \text{ Given that } |z_1| < 1 < |z_2|.$$

$$\text{Now, } \left| \frac{1 - z_1 \bar{z}_2}{z_1 - z_2} \right| < 1$$

$$\text{or } |1 - z_1 \bar{z}_2| < |z_1 - z_2|$$

$$\text{or } |1 - z_1 \bar{z}_2|^2 < |z_1 - z_2|^2$$

$$\text{or } (1 - z_1 \bar{z}_2)(\overline{1 - z_1 \bar{z}_2}) < (z_1 - z_2)(\overline{z_1 - z_2})$$

$$\text{or } (1 - z_1 \bar{z}_2)(1 - \bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$\text{or } 1 - z_1 \bar{z}_2 - \bar{z}_1 z_2 + z_1 \bar{z}_1 z_2 \bar{z}_2 < z_1 \bar{z}_1 - z_1 \bar{z}_2 - \bar{z}_1 z_2 + z_2 \bar{z}_2$$

$$\text{or } 1 + |z_1|^2 |z_2|^2 < |z_1|^2 + |z_2|^2$$

$$\text{or } (1 - |z_1|^2)(1 - |z_2|^2) < 0$$

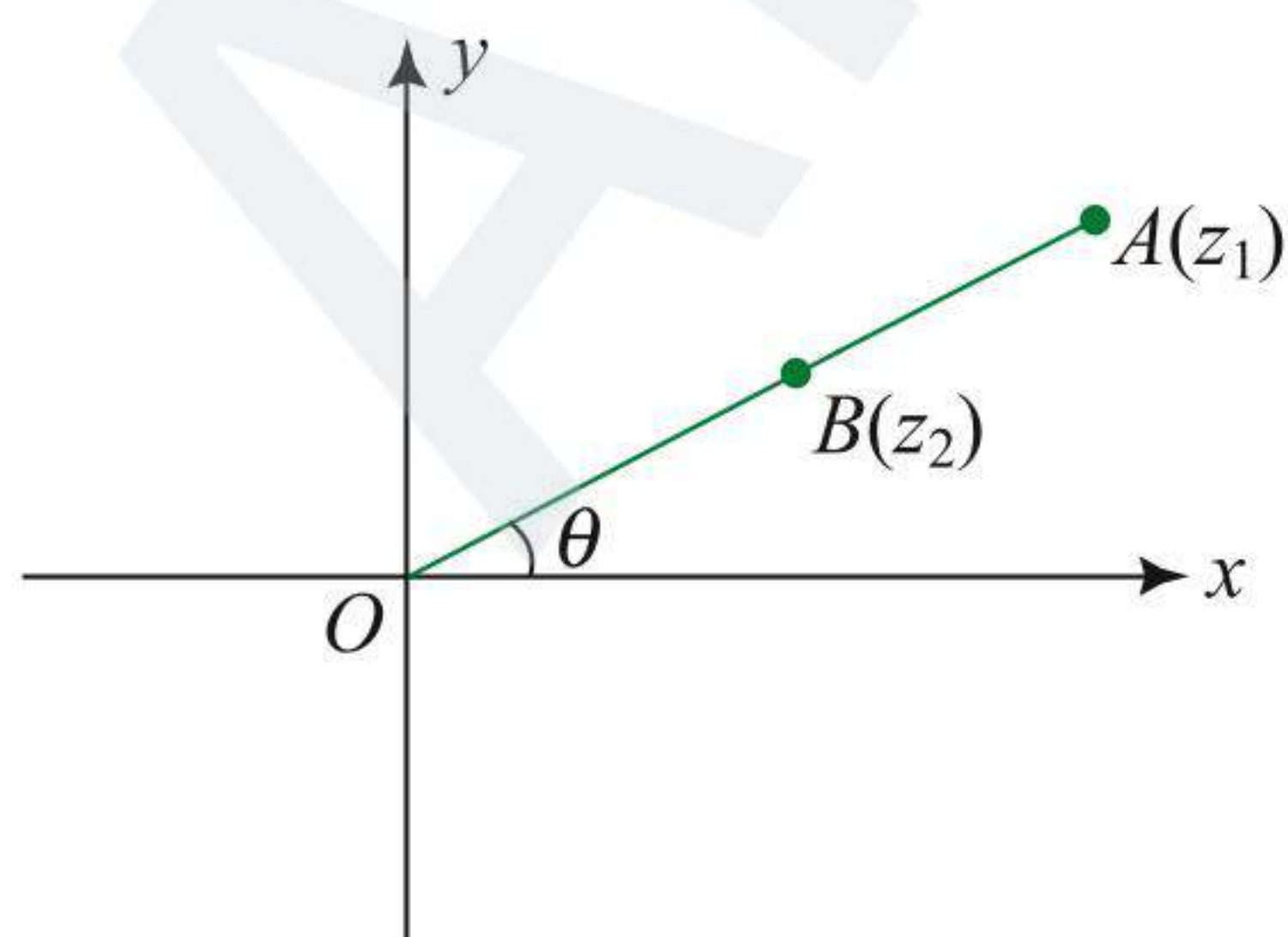
which is obviously true as

$$|z_1| < 1 < |z_2|$$

$$\Rightarrow |z_1|^2 < 1 < |z_2|^2$$

$$\Rightarrow (1 - |z_1|^2) > 0 \text{ and } (1 - |z_2|^2) < 0$$

$$5. (a) |z_1 - z_2| = |z_1| - |z_2|$$



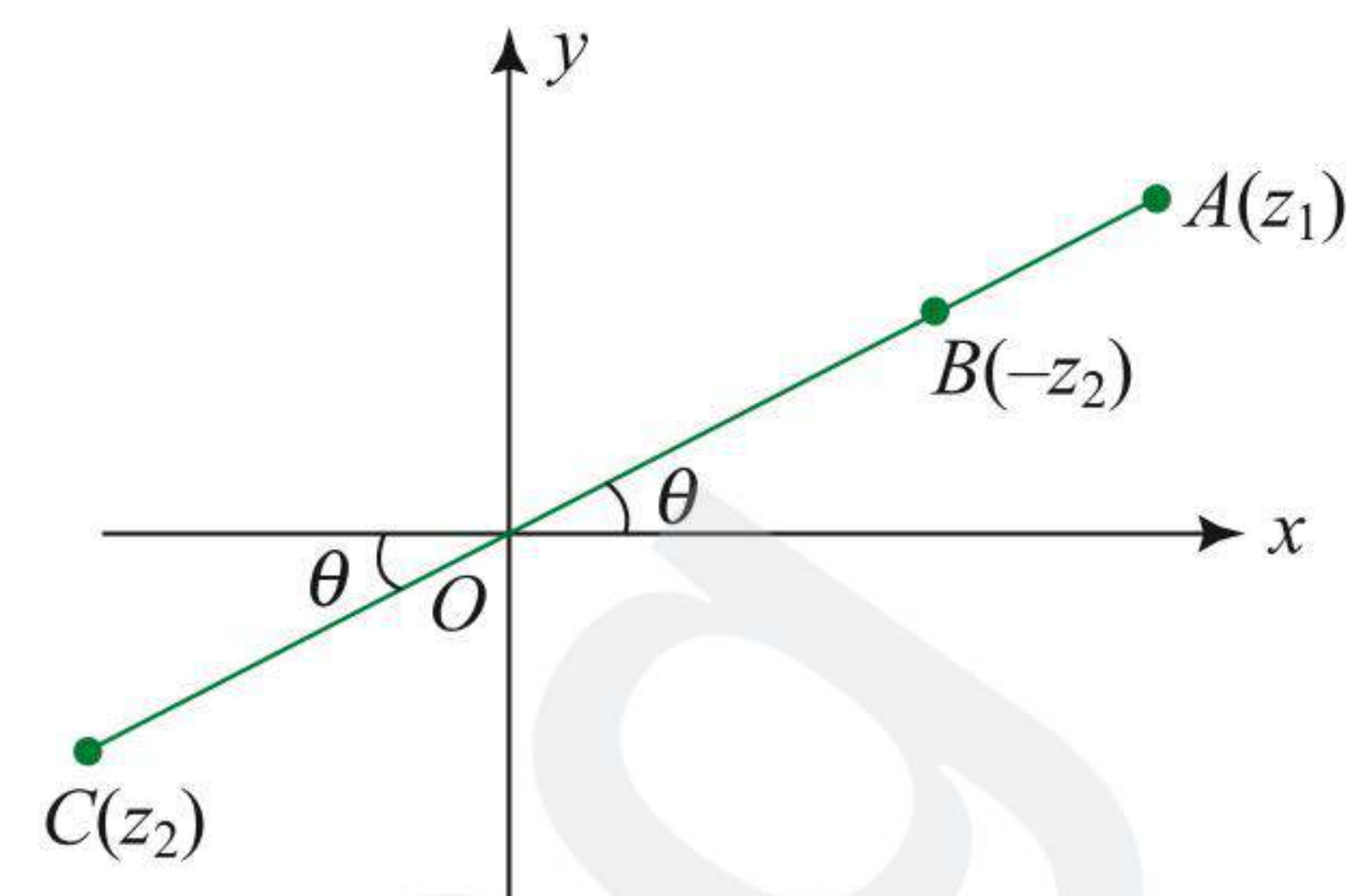
$$\Rightarrow AB = OA - OB \text{ as shown in figure}$$

$$\Rightarrow \text{Points } A(z_1), O(0), B(z_2) \text{ are collinear}$$

$$\Rightarrow \arg(z_1) = \arg(z_2) = \theta$$

$$(b) |z_1 + z_2| = |z_1| - |z_2|$$

$\Rightarrow$ Points $A(z_1), O(0), B(-z_2)$ are collinear as shown in figure



$$\Rightarrow \text{Points } A(z_1), O(0), C(z_2) \text{ are collinear}$$

$$\text{If } \arg(z_1) = \theta, \text{ then } \arg(z_2) = \theta - \pi$$

$$\Rightarrow \arg(z_1) - \arg(z_2) = \pi$$

$$6. \text{ We have, for } z \in \mathbb{C}$$

$$|2i| = |z + (2i - z)|$$

$$\leq |z| + |2i - z|$$

$$\Rightarrow 2 \leq |z| + |z - 2i|$$

Thus, minimum value of $|z| + |z - 2i|$ is 2.

$$7. |z + 1| = |z + 4 - 3|$$

$$= |(z + 4) + (-3)|$$

$$\leq |z + 4| + |-3|$$

$$= |z + 4| + 3$$

$$\leq 3 + 3 = 6$$

$$[\because |z + 4| \leq 3]$$

Hence, the greatest value of $|z + 1|$ is 6.

$$8. \left| Z + \frac{1}{Z} \right| = \left| Z - \left(-\frac{1}{Z} \right) \right| \geq |Z| - \left| -\frac{1}{Z} \right| \geq 3 - \frac{1}{3} = \frac{8}{3}$$

$$9. |a| = |b| = |c| = r$$

$$\text{Again } az^2 + bz = -c$$

$$\Rightarrow |c| = |-az^2 - bz| \leq |a||z|^2 + |b||z|$$

$$\Rightarrow r \leq r|z|^2 + r|z|$$

$$\Rightarrow |z|^2 + |z| - 1 \geq 0$$

$$\Rightarrow |z| \geq \frac{\sqrt{5} - 1}{2} \quad (1)$$

$$\text{Also from } az^2 = -bz - c,$$

$$|z|^2 - |z| - 1 \leq 0$$

$$\Rightarrow 0 < |z| \leq \frac{\sqrt{5} + 1}{2} \quad (2)$$

From (1) and (2),

$$\frac{\sqrt{5} - 1}{2} \leq |z| \leq \frac{\sqrt{5} + 1}{2}$$

$$10. |iz + 3 - 4i| \leq |iz| + |3 - 4i|$$

$$= |z| + 5 \leq 4 + 5 = 9$$

$$\text{Hence, } |z|_{\max} = 9.$$

$$11. z\bar{z} = 1 \text{ or } z = \frac{1}{z}$$

$$\text{Now, } z = (z_1 + z_2 + z_3 + \dots + z_n) \left(\frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right)$$

$$= (z_1 + z_2 + \dots + z_n)(\bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_n)$$

$$= (z_1 + z_2 + \dots + z_n)(\overline{z_1 + z_2 + \dots + z_n})$$

$$= |z_1 + z_2 + \dots + z_n|^2 \text{ which is real (a)}$$

$$\leq (|z_1| + |z_2| + \dots + |z_n|)^2 = n^2$$

$$\therefore 0 < z \leq n^2$$

Also, z is real number.

Exercise 3.9

$$1. \omega = \frac{z}{z - \frac{1}{3}i}$$

$$\text{or } |\omega| = \left| \frac{z}{z - \frac{1}{3}i} \right|$$

$$\text{or } |z| = |\omega| \left| z - \frac{1}{3}i \right|$$

$$\text{or } |z| = \left| z - \frac{1}{3}i \right| \quad [\because |\omega| = 1]$$

Hence, locus of z is perpendicular bisector of the line joining $0 + 0i$ and $0 + (1/3)i$. Hence z lies on a straight line.

$$2. \text{ Let } u = \frac{z-1}{e^{\theta i}} \Rightarrow \frac{e^{\theta i}}{z-1} = \frac{1}{u}$$

$$\text{Now } \left(u + \frac{1}{u} \right) - \left(\bar{u} + \frac{1}{\bar{u}} \right) = 0$$

$$\Rightarrow (u - \bar{u}) \left(1 - \frac{1}{u\bar{u}} \right) = 0$$

If u is not purely real, then $u\bar{u} = 1$

$$\Rightarrow \left| \frac{z-1}{e^{\theta i}} \right| = 1 \Rightarrow |z-1| = 1$$

Hence, z lies on circle having center at $1 + i0$ and radius 1

3. Consider the triangle formed by $A(Z)$, $B(Z_1)$ and $C(Z_2)$.

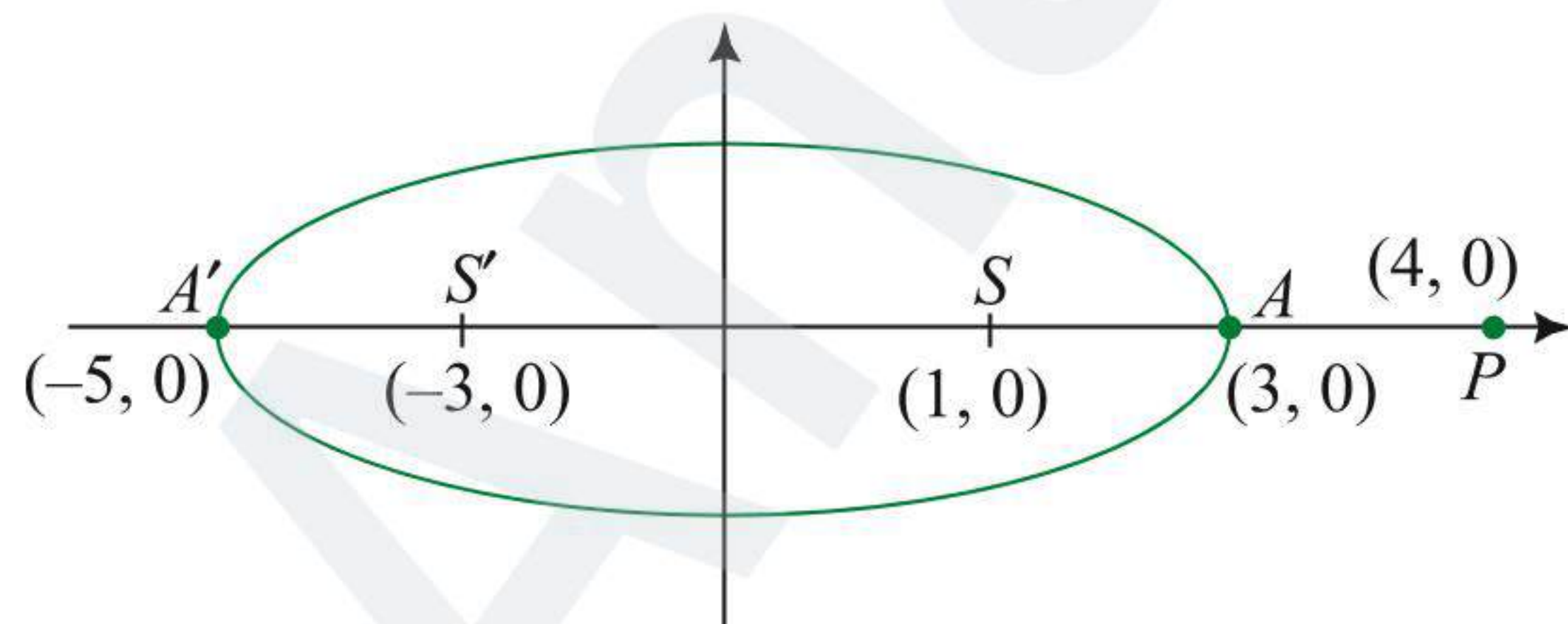
Let midpoint of BC be D having complex number $\frac{Z_1 + Z_2}{2}$.

By Apollonius theorem, we have

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

$$\begin{aligned} \therefore \left| Z - \frac{Z_1 + Z_2}{2} \right|^2 + \left| \frac{Z_1 - Z_2}{2} \right|^2 &= AD^2 + BD^2 \\ &= \frac{1}{2} AB^2 + \frac{1}{2} AC^2 \\ &= \frac{1}{2} |Z - Z_1|^2 + \frac{1}{2} |Z - Z_2|^2 \end{aligned}$$

4.



Given $|z-1| + |z+3| \leq 8$. Then z lies inside or on the ellipse whose foci are $(1, 0)$ and $(-3, 0)$ and vertices are $(-5, 0)$ and $(3, 0)$. Clearly, the minimum and maximum values of $|z-4|$ are 1 and 9, respectively, representing the distances PA and PA' . Thus, $1 \leq |z-4| \leq 9$.

$$5. \text{ Given } z = \frac{3}{2 + \cos \theta + i \sin \theta}$$

$$\Rightarrow \cos \theta + i \sin \theta = \frac{3}{z} - 2 = \frac{3-2z}{z}$$

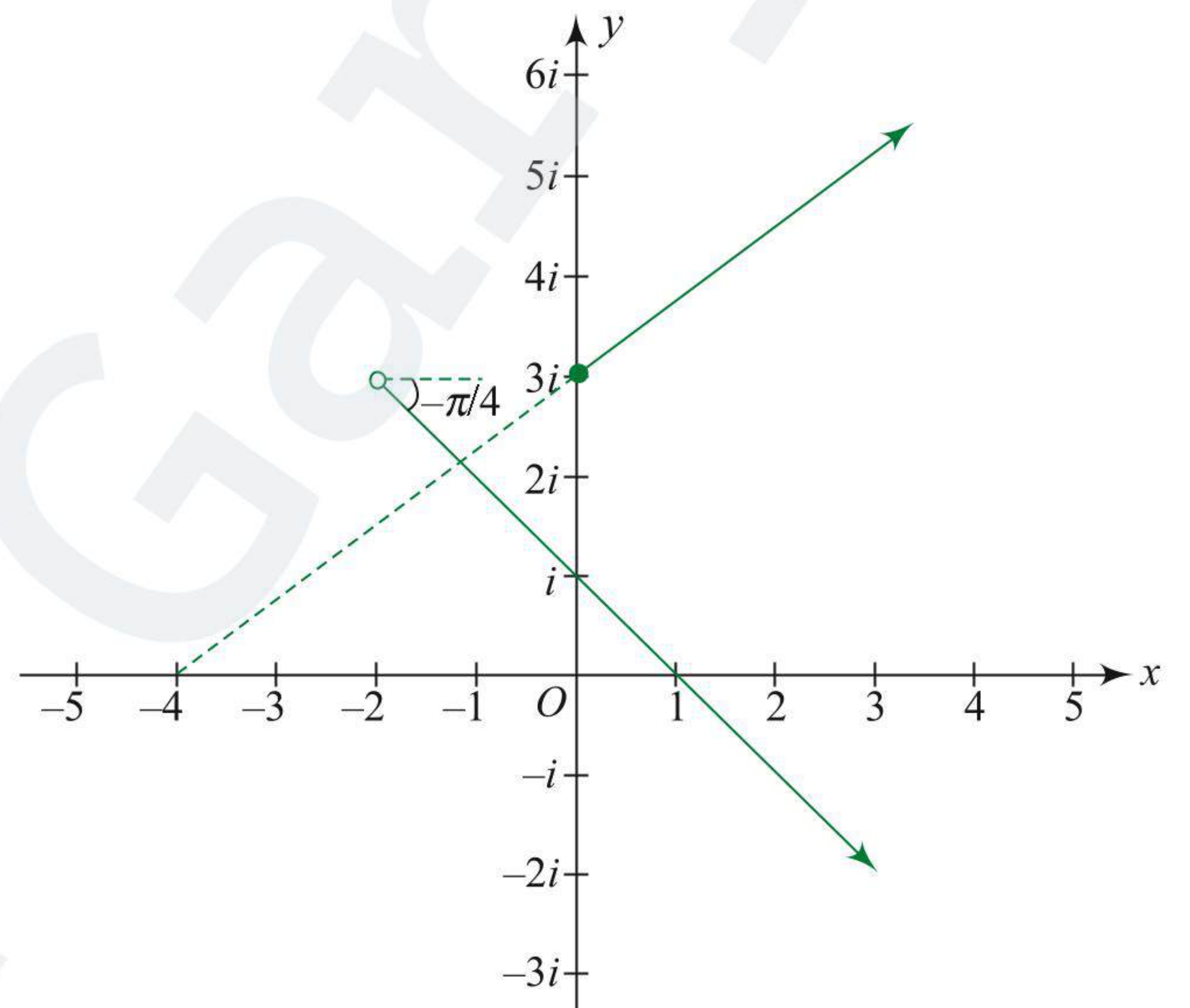
$$\Rightarrow 1 = \frac{|3-2z|}{|z|} \quad [\text{taking modulus}]$$

$$\Rightarrow \frac{\left| z - \frac{3}{2} \right|}{|z|} = \frac{1}{2}$$

Therefore, locus of z is circle.

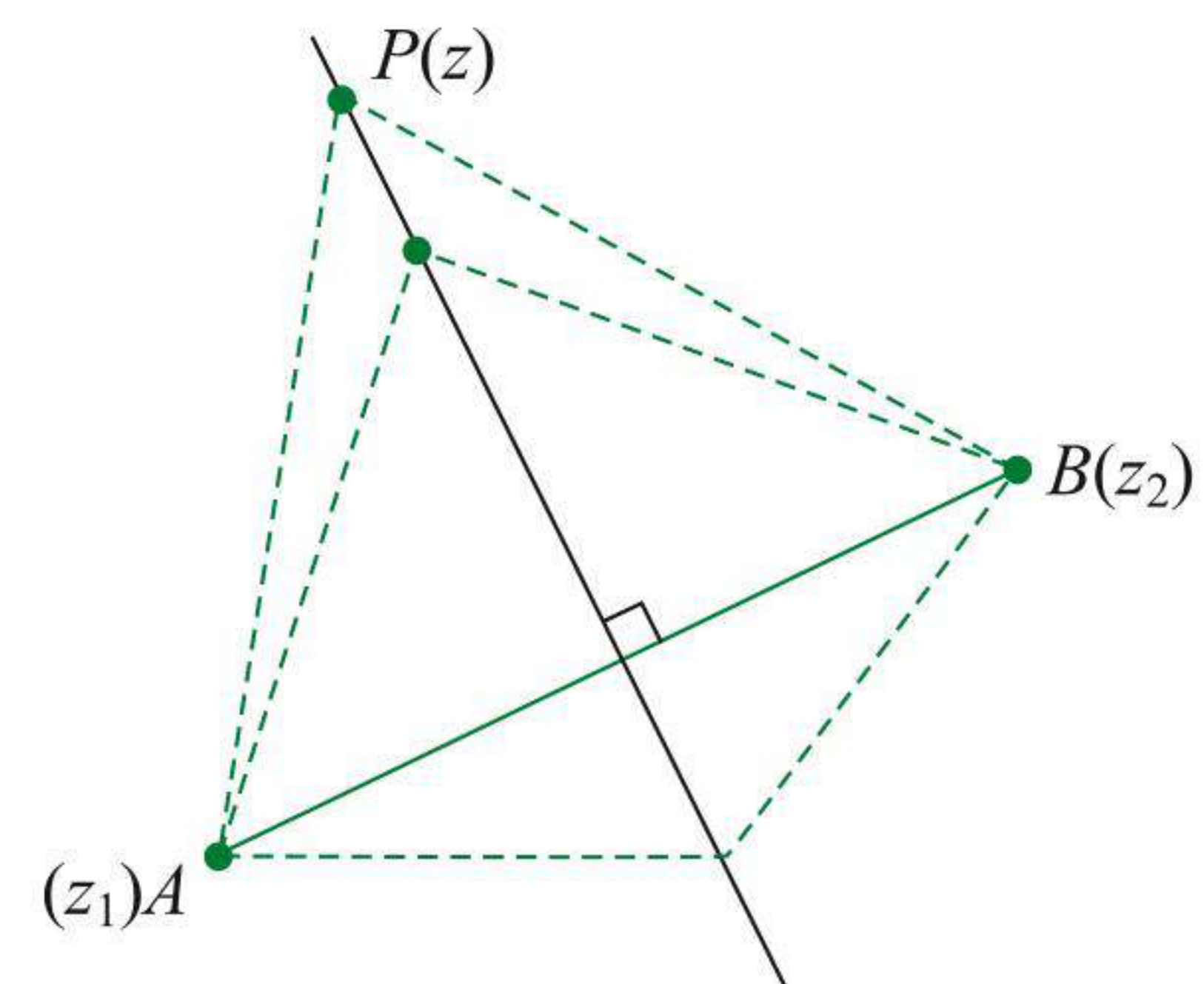
6. Equation $\arg(z - (-2 + 3i)) = -\pi/4$ represents the ray emanating from $(-2 + 3i)$ and making an angle of $-\pi/4$ with horizontal.

Equation $|z+4| - |z-3i| = 5$ or $|z - (-4)| - |z - 3i| = |3i+4|$ represents the ray emanating from $'3i'$ and moving to the right of $'3i'$.



Clearly, these two rays do not intersect. So, system of equations has no solution.

7. Let z be the complex number of any point on the perpendicular bisector.



Then $PA = PB$

$$\Rightarrow |z - z_1| = |z - z_2|$$

$$\Rightarrow |z - z_1|^2 = |z - z_2|^2$$

$$\Rightarrow |z|^2 + |z_1|^2 - z\bar{z}_1 - \bar{z}z_1 = |z|^2 + |z_2|^2 - z\bar{z}_2 - \bar{z}z_2$$

$$\Rightarrow z(\bar{z}_2 - \bar{z}_1) + \bar{z}(z_2 - z_1) + |z_1|^2 - |z_2|^2 = 0$$

8. We have $|z+1-i| = 1$ (1)

$$\text{and } w = \frac{z+i}{1-i}$$

$$\therefore w(1-i) = z+i$$

$$\Rightarrow z = w(1-i) - i$$

Putting the value of z in (1), we get

$$|w(1-i) - i + 1 - i| = 1$$

$$\Rightarrow \left| 1-i \right| \left| w + \frac{1-2i}{1-i} \right| = 1$$

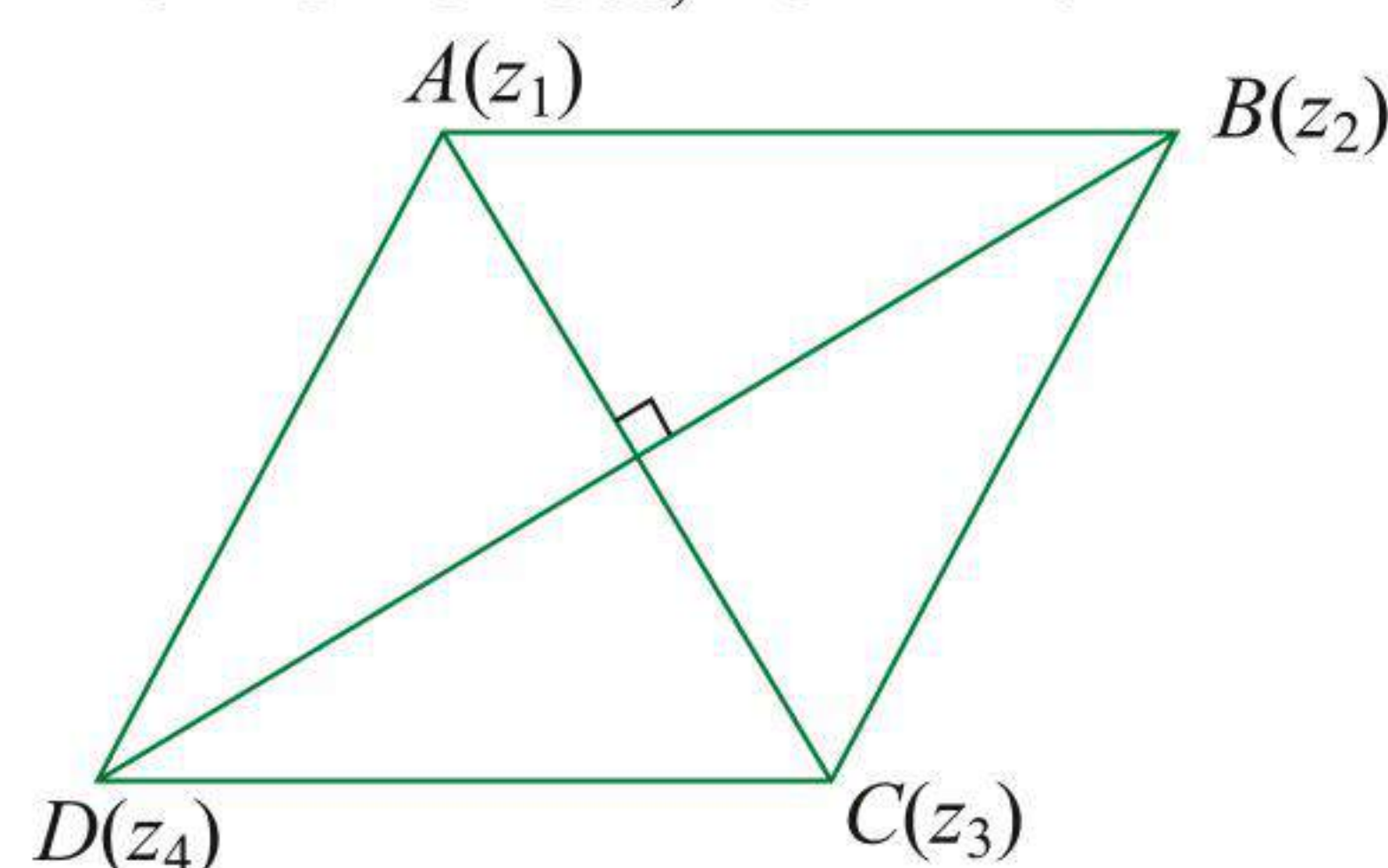
$$\Rightarrow \left| w + \frac{3-i}{2} \right| = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \left| w - \frac{-3+i}{2} \right| = \frac{1}{\sqrt{2}}$$

Therefore, locus of w is a circle having center at $(-3/2, 1/2)$ and radius $\frac{1}{\sqrt{2}}$.

Exercise 3.10

1. Since $ABCD$ is rhombus, $AC \perp BD$.



$$\arg\left(\frac{z_3 - z_1}{z_4 - z_2}\right) = \pm \frac{\pi}{2}$$

Therefore, $\frac{z_3 - z_1}{z_4 - z_2}$ is purely imaginary.

$$\therefore \operatorname{Re}\left(\frac{z_3 - z_1}{z_4 - z_2}\right) = 0$$

2. Given that points z , i and iz are collinear.

$$\therefore \arg\left(\frac{z-i}{iz-i}\right) = 0, \pi$$

So, $\frac{z-i}{iz-i}$ is purely real.

$$\Rightarrow \frac{z-i}{iz-i} = \frac{\bar{z}+i}{-i\bar{z}+i}$$

$$\Rightarrow \frac{z-i}{z-1} = \frac{\bar{z}+i}{-\bar{z}+1}$$

$$\Rightarrow 2z\bar{z} + iz - i\bar{z} - \bar{z} - z = 0$$

$$\Rightarrow 2(x^2 + y^2) - 2x - 2y = 0$$

$$\Rightarrow x^2 + y^2 - x - y = 0$$

3. For circle $|z|=2$, points $z=2$ and $z=-2$ are end points of diameter.

$$\Rightarrow \arg\left(\frac{z-2}{z+2}\right) = \pm \frac{\pi}{2}$$

$$\Rightarrow \arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \pm \frac{\pi}{2}$$

Therefore, z_1, z_2 , and z_3 are the vertices of a right angled triangle.

4. Since $AD \perp BC$

$$\Rightarrow \arg\left(\frac{\alpha - z}{\gamma - \beta}\right) = \frac{\pi}{2}$$

So, $\frac{z-\alpha}{\beta-\gamma}$ is purely imaginary.

$$\Rightarrow \operatorname{Re}\left(\frac{z-\alpha}{\beta-\gamma}\right) = 0$$

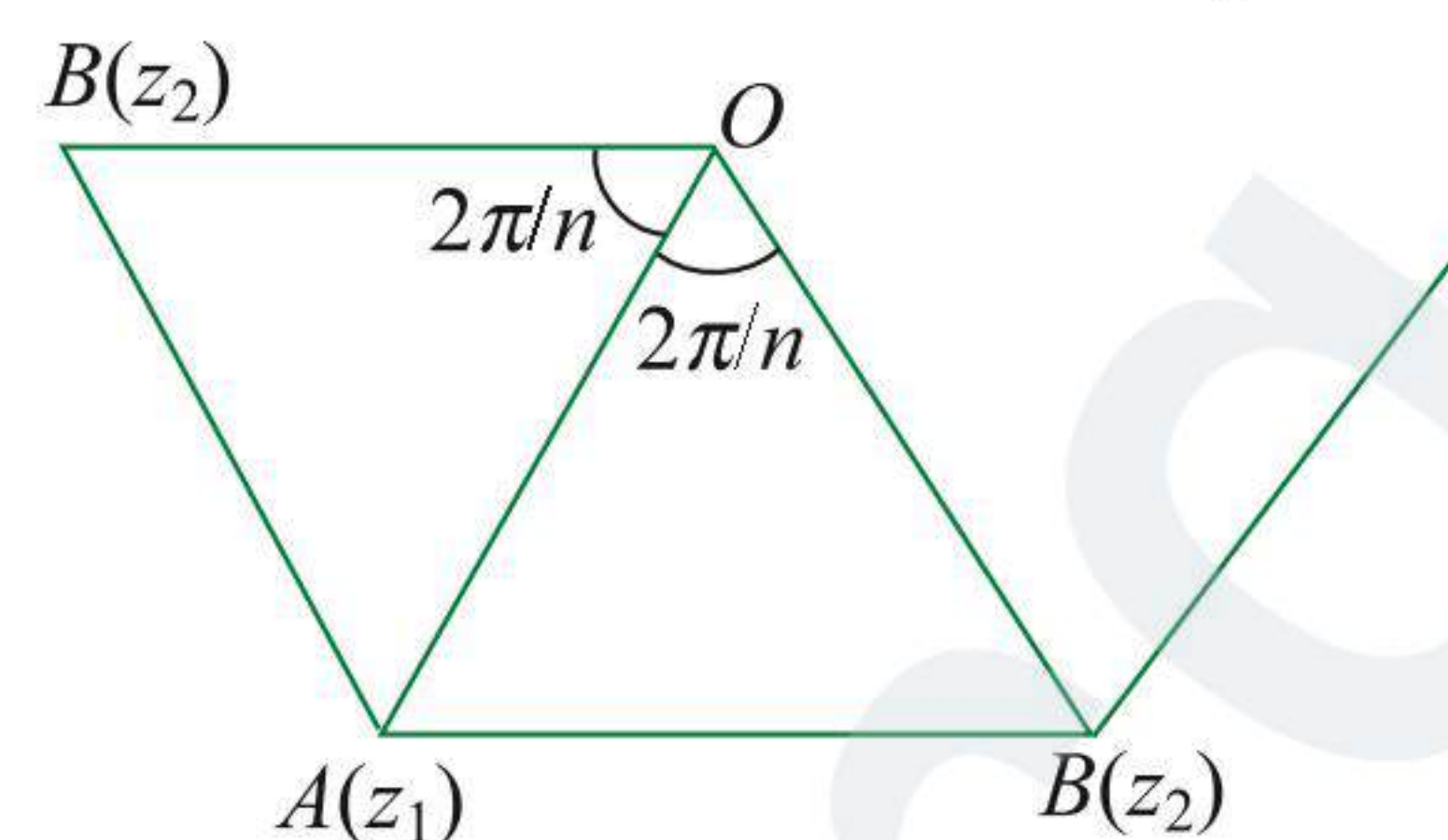
5. We know that if z_1, z_2 and z_3 form an equilateral triangle, then

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Putting $z_3 = 0$, we get

$$z_1^2 + z_2^2 - z_1z_2 = 0$$

6. Let A be the vertex with affix z_1 .



z_2 can be obtained by rotating z_1 through $2\pi/n$ either in clockwise or in anticlockwise direction. Therefore,

$$\frac{z_2}{z_1} = \left| \frac{z_2}{z_1} \right| e^{\pm \frac{i2\pi}{n}}$$

$$\text{or } z_2 = z_1 e^{\pm \frac{i2\pi}{n}} \quad [\because |z_2| = |z_1|]$$

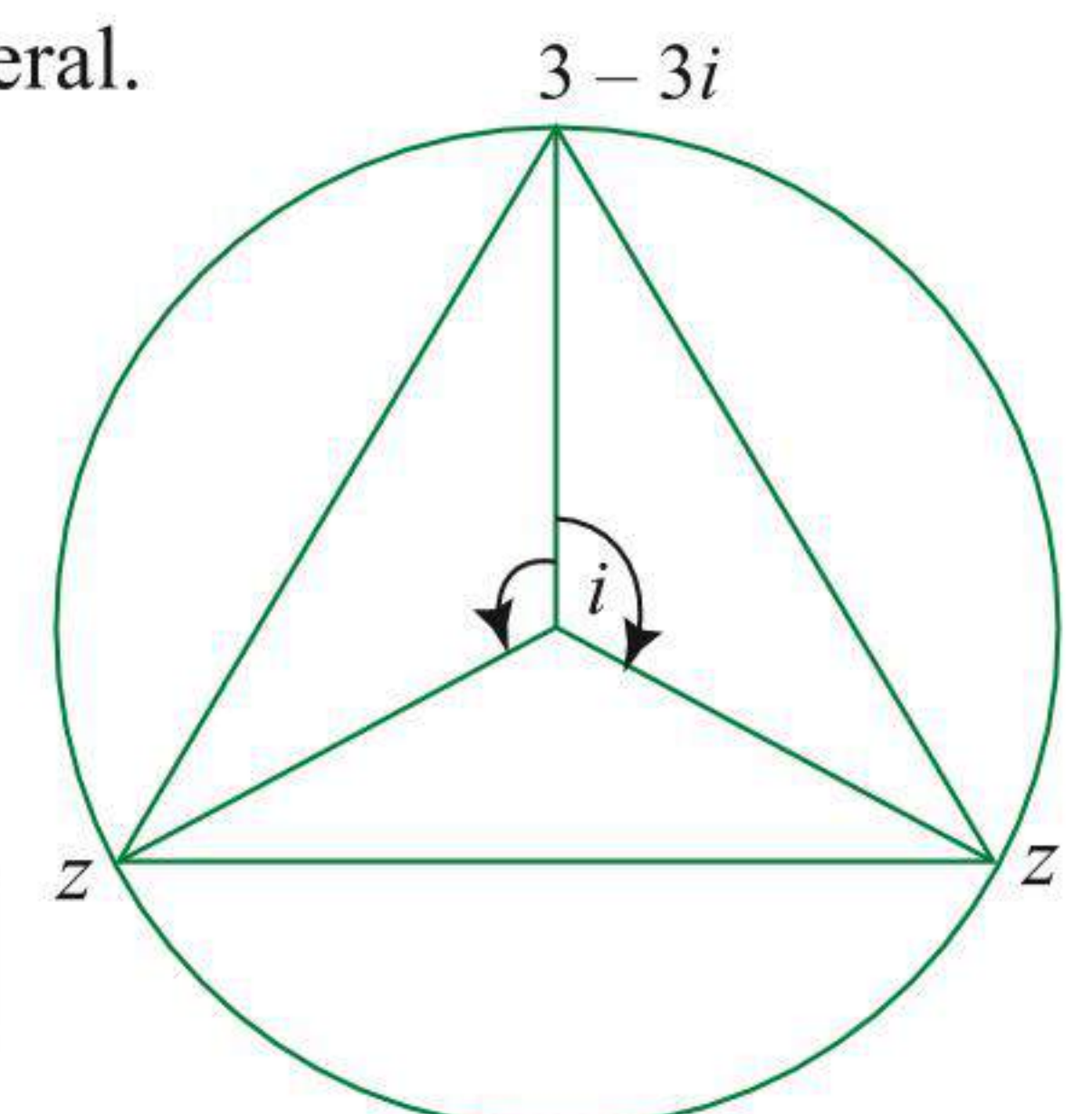
7. Inscribed triangle is clearly equilateral.

Let the other vertex be z .

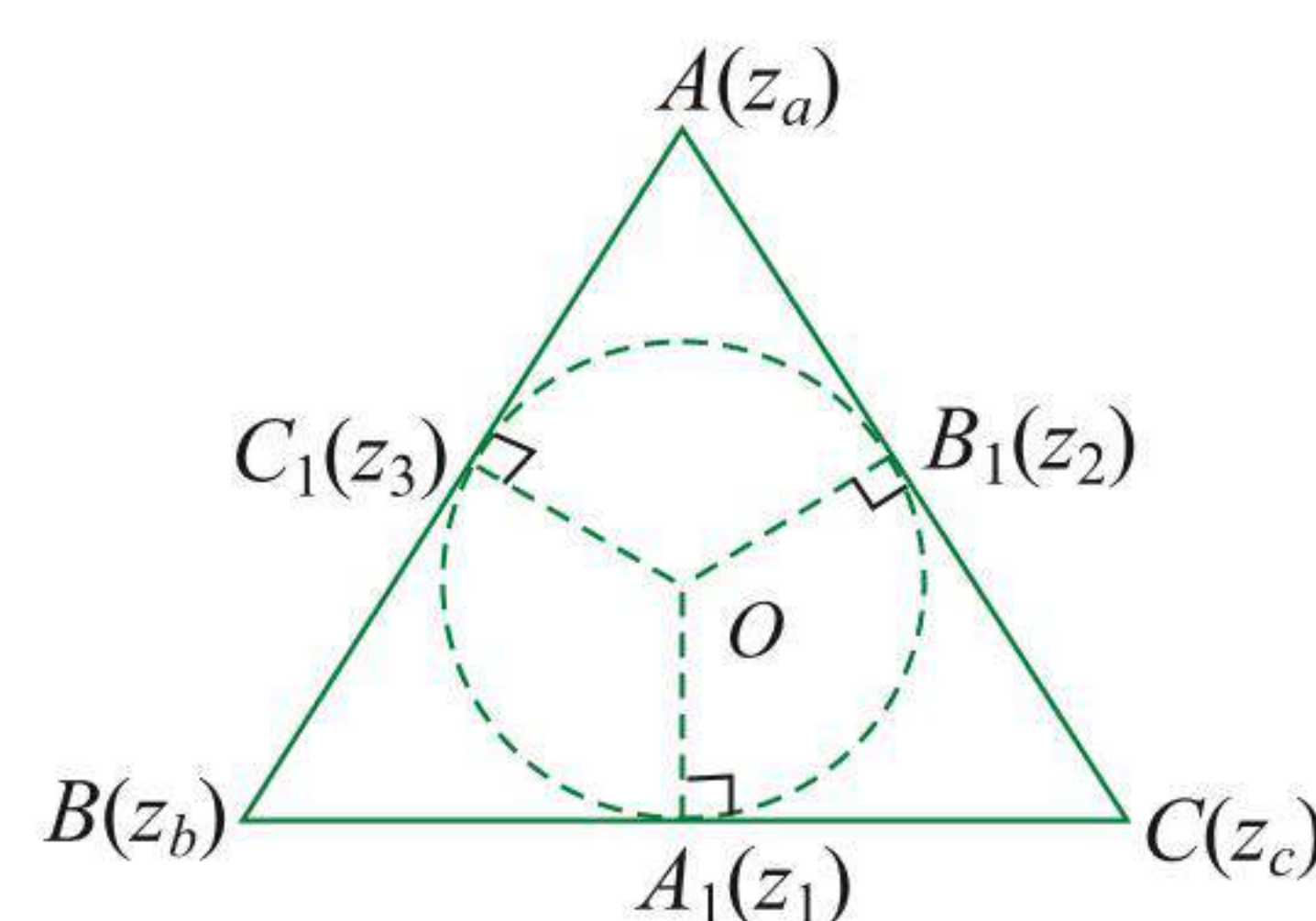
$$\therefore \frac{z-i}{3-3i-i} = e^{\pm \frac{i2\pi}{3}}$$

$$\text{or } z-i = (3-4i)e^{\pm \frac{i2\pi}{3}}$$

$$\text{or } z = i + (3-4i) \left(-\frac{1}{2} \pm i \frac{\sqrt{3}}{2} \right)$$



8. Let $A(z_a), B(z_b)$ and $C(z_c)$ be the vertices of the triangle.



Considering the rotation about points A_1 and B_1 , respectively, we get

$$\frac{z_c - z_1}{0 - z_1} = \frac{|z_c - z_1|}{r} e^{-i\pi/2} \quad \text{and} \quad \frac{z_c - z_2}{0 - z_2} = \frac{|z_c - z_2|}{r} e^{i\pi/2}$$

$$\therefore \frac{(z_c - z_1)z_2}{z_1(z_c - z_2)} = e^{-i\pi} = -1$$

$$\Rightarrow (z_c - z_1)z_2 = -z_1(z_c - z_2)$$

$$\text{or } z_c = \frac{2z_1z_2}{(z_1 + z_2)}$$

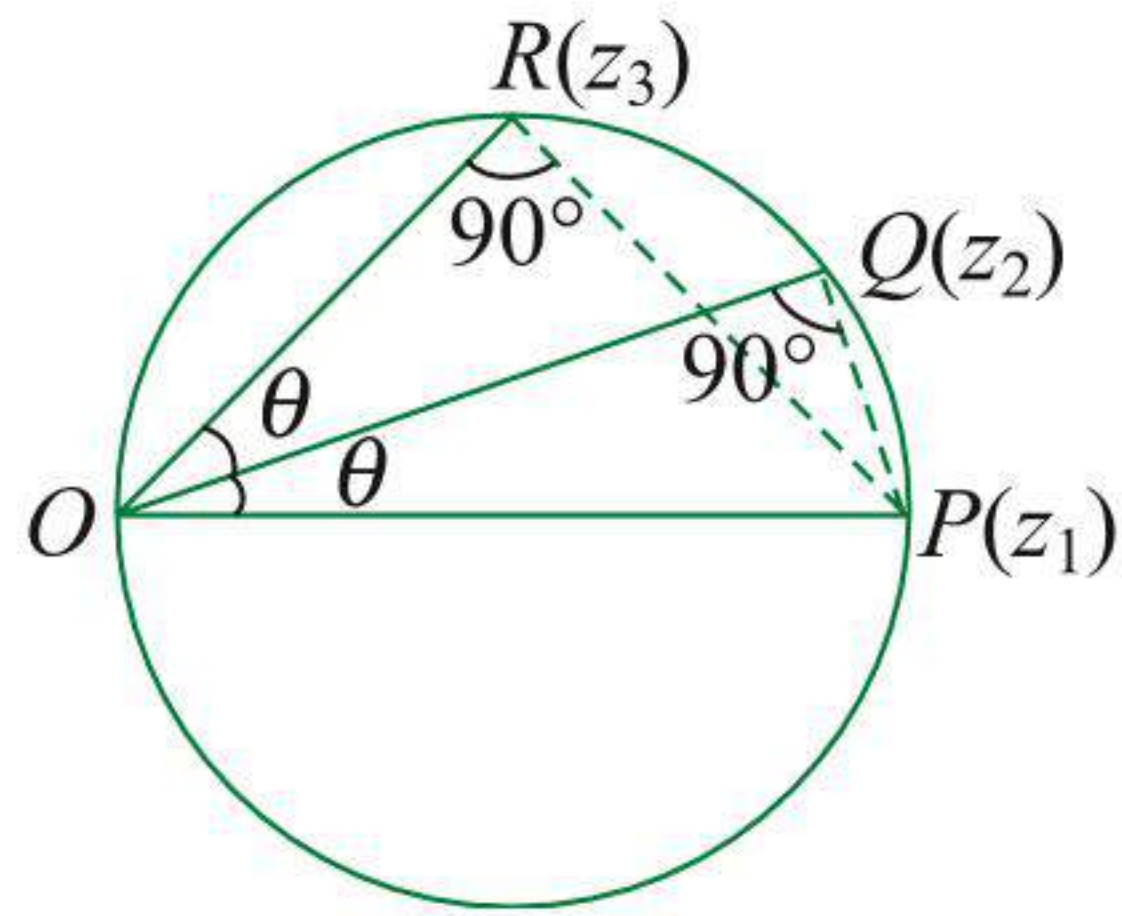
Similarly,

$$z_a = \frac{2z_2z_3}{(z_2 + z_3)} \quad \text{and} \quad z_b = \frac{2z_1z_3}{(z_1 + z_3)}$$

$$9. \quad z_2 = \frac{OQ}{OP} z_1 e^{i\theta} = \cos \theta z_1 e^{i\theta} \quad (1)$$

$$z_3 = \frac{OR}{OP} z_1 e^{i2\theta} = \cos 2\theta z_1 e^{i2\theta} \quad (2)$$

From (1), $z_2^2 = \cos^2 \theta z_1^2 e^{2i\theta}$

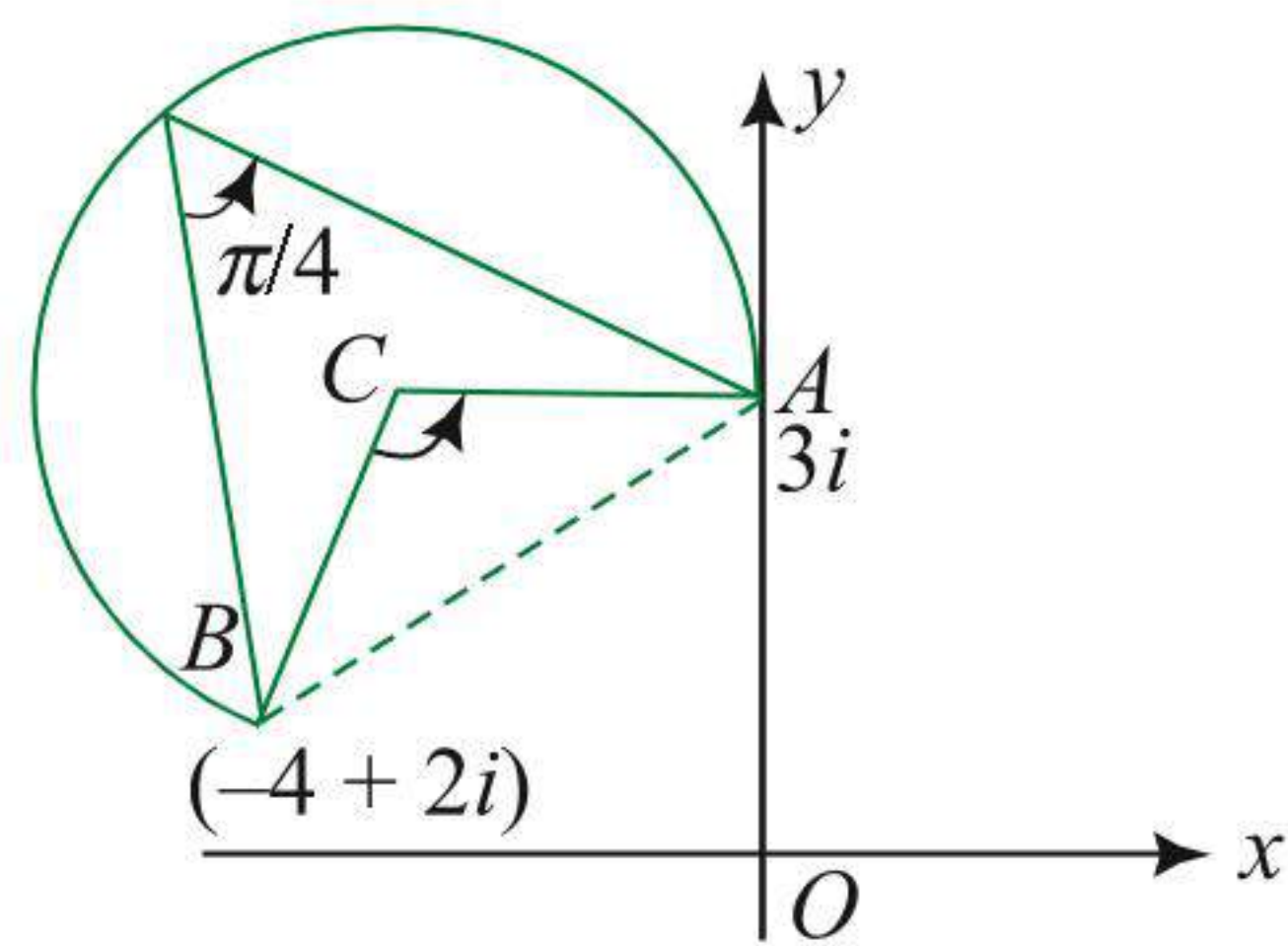


Dividing (3) by (2), we get

$$\frac{z_2^2}{z_3} = \frac{\cos^2 \theta z_1}{\cos 2\theta}$$

Hence, $z_2^2 \cos 2\theta = z_1 z_3 \cos^2 \theta$

10. If C is the center of the arc, then $\angle BCA = \pi/2$.



Let C be z_c . Then,

$$\frac{z_c - 3i}{z_c - 2i + 4} = e^{i\pi/2} = i$$

$$z_c = 3i + i(z_c - 2i + 4)$$

$$\therefore z_c = \frac{7i + 2}{(1 - i)} = \frac{1}{2}(9i - 5)$$

Exercise 3.11

1. α is a complex fifth roots of unity.

$$\therefore 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$$

Now, given that

$$(1 + \alpha + \alpha^2 + \alpha^3)^{2005} = p + q\alpha + r\alpha^2 + s\alpha^3$$

$$\Rightarrow (-\alpha^4)^{2005} = p + q\alpha + r\alpha^2 + s\alpha^3$$

$$\Rightarrow -\alpha^{8020} = p + q\alpha + r\alpha^2 + s\alpha^3$$

$$\Rightarrow -(\alpha^5)^{1604} = p + q\alpha + r\alpha^2 + s\alpha^3$$

$$\Rightarrow -1 = p + q\alpha + r\alpha^2 + s\alpha^3$$

$$\Rightarrow p = -1, q = r = s = 0$$

$$\therefore p + q + r + s = -1$$

2. We have $z^{15} = 1$

$$z = 1^{1/15} = e^{i \frac{2r\pi}{15}}, r = 0, 1, 2, \dots, 14$$

$$\text{or } z = e^{\pm i \frac{2r\pi}{15}}, r = 0, 1, 2, \dots, 7.$$

The complex numbers having their arguments between $-\frac{\pi}{2}$ and

$$\frac{\pi}{2} \text{ are } e^0, e^{\pm i \frac{2\pi}{15}}, e^{\pm i \frac{4\pi}{15}}, e^{\pm i \frac{6\pi}{15}}.$$

Hence number of solutions is 7.

3. Let $z = \sqrt[7]{-1}$. Then

$$z^7 = -1$$

$$\therefore z^{86} + z^{175} + z^{289}$$

$$= (z^7)^{12} \times z^2 + (z^7)^{25} + (z^7)^{41} \times z^2$$

$$= (-1)^{12} \times z^2 + (-1)^{25} + (-1)^{41} \times z^2$$

$$= z^2 - 1 - z^2 = -1$$

(3)

4. As α is the fifth nonreal root of unity, we have

$$\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$$

β is the fourth nonreal root of unity. Therefore,

$$\beta^3 + \beta^2 + \beta + 1 = 0$$

Now, $(1 + \alpha)(1 + \alpha^2)(1 + \alpha^4)(1 + \beta)(1 + \beta^2)(1 + \beta^3)$

$$= (1 + \alpha + \alpha^2 + \alpha^3)(1 + \alpha^4)(1 + \beta + \beta^2 + \beta^3)(1 + \beta^3)$$

$$= 0 \quad (\because 1 + \beta + \beta^2 + \beta^3 = 0)$$

5. $x^6 = -64$

$$\Rightarrow \left(\frac{x}{2i}\right)^6 = 1$$

$$\Rightarrow \frac{x}{2i} = \cos\left(\frac{2n\pi}{6}\right) + i\sin\left(\frac{2n\pi}{6}\right), \text{ where } n = 0, 1, 2, 3, 4, 5$$

$$\Rightarrow x = 2i\cos\left(\frac{n\pi}{3}\right) - 2\sin\left(\frac{n\pi}{3}\right), \text{ where } n = 0, 1, 2, 3, 4, 5$$

For $n = 4$ and 5 we have positive real part. Hence, the required product is

$$\left[2i\cos\left(\frac{4\pi}{3}\right) - 2\sin\left(\frac{4\pi}{3}\right)\right] \left[2i\cos\left(\frac{5\pi}{3}\right) - 2\sin\left(\frac{5\pi}{3}\right)\right]$$

$$= 4 \left[-i\frac{1}{2} + \frac{\sqrt{3}}{2}\right] \left[i\frac{1}{2} + \frac{\sqrt{3}}{2}\right]$$

$$= 4$$

$$6. E = \frac{1}{z_1 - 1} + \frac{1}{z_2 - 1} + \dots + \frac{1}{z_{50} - 1}$$

where $z_1, z_2, \dots, z_{50}$ are the roots of the equation $z^{51} - 1 = 0$ other than 1. Let

$$1 + z + z^2 + \dots + z^{50} = (z - z_1)(z - z_2) \dots (z - z_{50})$$

$$\Rightarrow \log(1 + z + z^2 + \dots + z^{50}) = \log[(z - z_1)(z - z_2) \dots (z - z_{50})]$$

Differentiating both sides w.r.t. z and putting $z = 1$,

$$\frac{1 + 2z + 3z^2 + \dots + 50z^{49}}{1 + z + z^2 + \dots + z^{50}}$$

$$= \frac{1}{z - z_1} + \frac{1}{z - z_2} + \dots + \frac{1}{z - z_{50}}$$

$$\Rightarrow \frac{50 \times 51}{2 \times 51} = - \left[\frac{1}{z_1 - 1} + \frac{1}{z_2 - 1} + \dots + \frac{1}{z_{50} - 1} \right]$$

$$\therefore \sum \frac{1}{z_r - 1} = -25$$

Exercises

Single Correct Answer Type

1. (2) Verify by selecting particular values of a and b .

Let $a = -9$ and $b = 4$. Then

$$\sqrt{a}\sqrt{b} = \sqrt{-9}\sqrt{4} = (3i)(2) = 6i$$

From option (1), we have

$$-\sqrt{|a|b} = -\sqrt{|-9| \times 4} = -\sqrt{36} = -6$$

From option (2), we have

$$\sqrt{|a|b}i = \sqrt{|-9| \times 4} i = 6i$$

2. (4) $10z^2 - 3iz - k = 0$

$$\Rightarrow z = \frac{3i \pm \sqrt{-9 + 40k}}{20}$$

Now, $D = -9 + 40k$. If $k = 1$, then $D = 31$. So (1) is false.

If k is a negative real number, then D is a negative real number. So (4) is true.

If $k = i$, then $D = -9 + 40i = 16 + 40i - 25 = (4 + 5i)^2$, and the roots are $(1/5) + (2/5)i$ and $-(1/5 - 1/10)i$. So (3) is false.

If $k = 0$ (which is a complex number), then the roots are 0 and $(3/10)i$. So (2) is false.

3. (4) Let $z = x + iy$, so that $\bar{z} = x - iy$.

$$\therefore z^2 + \bar{z} = 0$$

$$\Rightarrow (x^2 - y^2 + x) + i(2xy - y) = 0$$

Equating real and imaginary parts, we get

$$x^2 - y^2 + x = 0 \quad (1)$$

$$\text{and } 2xy - y = 0 \Rightarrow y = 0 \text{ or } x = \frac{1}{2}$$

If $y = 0$, then (1) gives $x^2 + x = 0$

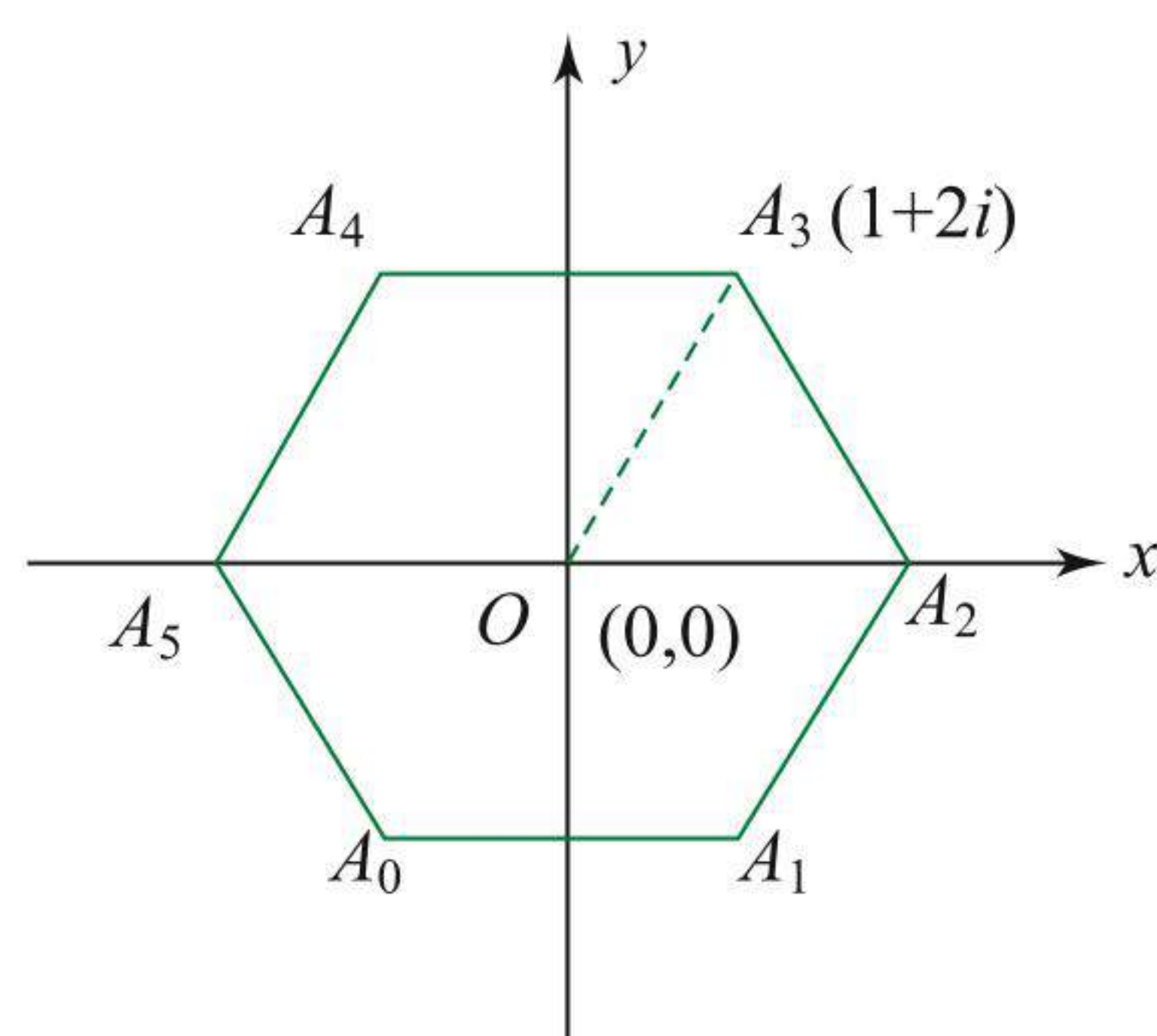
$$\Rightarrow x = 0 \text{ or } x = -1$$

If $x = 1/2$, then from Eq. (1),

$$y^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \text{ or } y = \pm \frac{\sqrt{3}}{2}$$

Hence, there are four solutions in all.

4. (4)



Let the vertices be $z_0, z_1, \dots, z_5$ w.r.t. center O at origin and $|z_0| = \sqrt{5}$.

Now $\triangle OA_2A_3$ is equilateral

$$\Rightarrow OA_2 = OA_3 = A_2A_3 = \sqrt{5}$$

$$\text{Perimeter} = 6\sqrt{5}$$

5. (3) Observing carefully the system of equations, we find

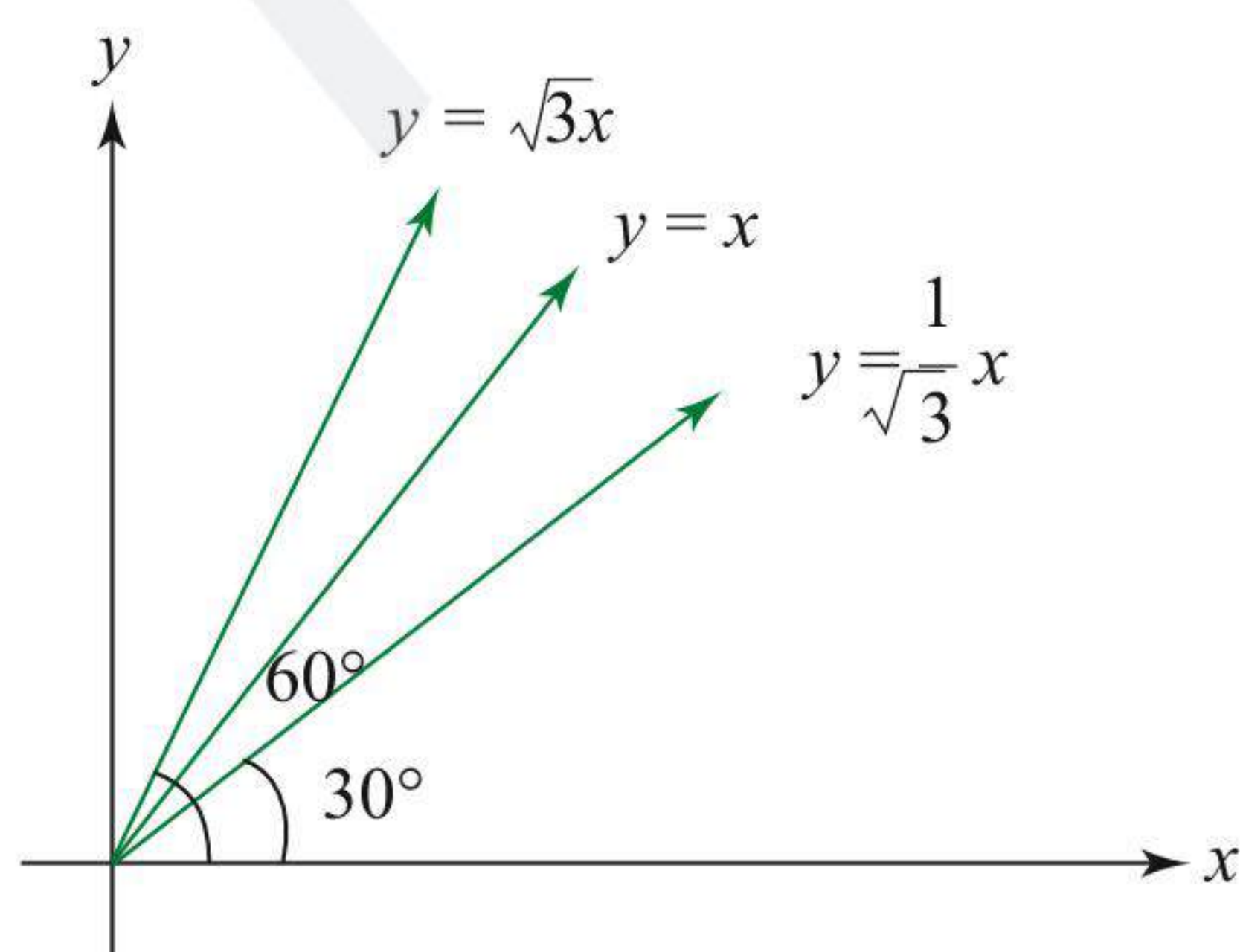
$$\frac{1+i}{2i} = \frac{1-i}{2} = \frac{1}{1+i}$$

Hence, there are infinite number of solutions.

6. (2) Note that $z_1 = 3 + \sqrt{3}i$ lies on the line $y = (1/\sqrt{3})x$

and $z_2 = 2\sqrt{3} + 6i$ lies on the line $y = \sqrt{3}x$.

Hence, $z = 5 + 5i$ will only lie on the bisector of z_1 and z_2 , i.e., $y = x$.



7. (4) Let $f(x) = x^6 + 4x^5 + 3x^4 + 2x^3 + x + 1$. Hence,

$$\begin{aligned} f(\omega) &= \omega^6 + 4\omega^5 + 3\omega^4 + 2\omega^3 + \omega + 1 \\ &= 1 + 4\omega^2 + 3\omega + 2 + \omega + 1 \\ &= 4(\omega^2 + \omega + 1) \\ &= 0 \end{aligned}$$

Hence, $f(x)$ is divisible by $x - \omega$. Then $f(x)$ is also divisible by $x - \omega^2$ (as complex roots occur in conjugate pairs).

$$\begin{aligned} f(-\omega) &= (-\omega)^6 + 4(-\omega)^5 + 3(-\omega)^4 + 2(-\omega)^3 + (-\omega) + 1 \\ &= \omega^6 - 4\omega^5 + 3\omega^4 - 2\omega^3 - \omega + 1 \\ &= 1 - 4\omega^2 + 3\omega - 2 - \omega + 1 \\ &\neq 0 \end{aligned}$$

8. (2) $f(z) = g(z)(z - i)(z + i) + az + b$; $a, b \in \mathbb{C}$

Given, $f(i) = i$

$$\Rightarrow ai + b = i \quad (1)$$

and $f(-i) = 1 + i$

$$\Rightarrow a(-i) + b = 1 + i \quad (2)$$

From (1) and (2), we have

$$a = \frac{i}{2}, b = \frac{1}{2} + i$$

Hence, the required remainder is $az + b = (1/2)iz + (1/2) + i$.

9. (4) Let $z_1 = \sin x + i \cos 2x$ and $z_2 = \cos x - i \sin 2x$.

Then $\bar{z}_1 = z_2$

$$\Rightarrow \sin x - i \cos 2x = \cos x - i \sin 2x$$

$$\Rightarrow \sin x = \cos x \text{ and } \cos 2x = \sin 2x$$

$$\Rightarrow \tan x = 1 \text{ and } \tan 2x = 1$$

$$\Rightarrow x = \frac{\pi}{4} \text{ and } x = \frac{\pi}{8}$$

This is not possible.

Hence, there is no value of x .

10. (2) Let xi be the root where $x \neq 0$ and $x \in \mathbb{R}$

$$x^4 - a_1x^3i - a_2x^2 + a_3xi + a_4 = 0$$

$$\Rightarrow x^4 - a_2x^2 + a_4 = 0 \quad (1)$$

$$\text{and } a_1x^3 - a_3x = 0 \quad (2)$$

From Eq. (2),

$$a_1x^2 - a_3 = 0$$

$$\Rightarrow x^2 = a_3/a_1 \text{ (as } x \neq 0)$$

Putting the value of x^2 in Eq. (1), we get

$$\frac{a_3^2}{a_1^2} - \frac{a_2a_3}{a_1} + a_4 = 0$$

$$\Rightarrow a_3^2 + a_4a_1^2 = a_1a_2a_3$$

$$\Rightarrow \frac{a_3}{a_1a_2} + \frac{a_1a_4}{a_2a_3} = 1 \text{ (dividing by } a_1a_2a_3)$$

11. (3) We have,

$$z_1(z_1^2 - 3z_2^2) = 2 \quad (1)$$

$$z_2(3z_1^2 - z_2^2) = 11 \quad (2)$$

Multiplying (2) by i and adding it to (1), we get

$$z_1^3 - 3z_2^2z_1 + i(3z_1^2z_2 - z_2^3) = 2 + 11i$$

$$\Rightarrow (z_1 + iz_2)^3 = 2 + 11i \quad (3)$$

Multiplying (2) by i and subtracting it from (1), we get

$$z_1^3 - 3z_2^2z_1 - i(3z_1^2z_2 - z_2^3) = 2 - 11i$$

$$\Rightarrow (z_1 - iz_2)^3 = 2 - 11i \quad (4)$$

Multiplying (3) and (4), we get

$$(z_1^2 + z_2^2)^3 = 2^2 - 121i^2 = 4 + 121 = 125$$

$$\Rightarrow z_1^2 + z_2^2 = 5$$

12. (3) Given that $a^2 + b^2 = 1$. Therefore,

$$\begin{aligned}\frac{1+b+ia}{1+b-ia} &= \frac{(1+b+ia)(1+b+ia)}{(1+b-ia)(1+b+ia)} \\ &= \frac{(1+b)^2 - a^2 + 2ia(1+b)}{1+b^2+2b+a^2} \\ &= \frac{(1-a^2) + 2b + b^2 + 2ia(1+b)}{2(1+b)} \\ &= \frac{2b^2 + 2b + 2ia(1+b)}{2(1+b)} = b + ia\end{aligned}$$

$$\begin{aligned}13. (1) \quad \frac{1+iz}{1-iz} &= \frac{1+i(b+ic)/(1+a)}{1-i(b+ic)/(1+a)} \\ &= \frac{1+a-c+ib}{1+a+c-ib} \\ &= \frac{(1+a-c+ib)(1+a+c+ib)}{(1+a+c)^2 + b^2} \\ &= \frac{1+2a+a^2-b^2-c^2+2ib+2iab}{1+a^2+c^2+b^2+2ac+2(a+c)} \\ &= \frac{2a+2a^2+2ib+2iab}{2+2ac+2(a+c)} \quad (\because a^2+b^2+c^2=1) \\ &= \frac{a+a^2+ib+iab}{1+ac+(a+c)} \\ &= \frac{a(a+1)+ib(a+1)}{(a+1)(c+1)} = \frac{a+ib}{c+1}\end{aligned}$$

14. (1) Let α be the real root and $i\beta$ be the imaginary root of the given equation. Then

$$\begin{aligned}\alpha + i\beta &= -a \\ \Rightarrow \alpha - i\beta &= -\bar{a} \\ \text{So, } 2\alpha &= -(a + \bar{a}) \text{ and } 2i\beta = -(a - \bar{a})\end{aligned}$$

Multiplying these, we get

$$4i\alpha\beta = a^2 - (\bar{a})^2$$

$$\therefore 4b = a^2 - (\bar{a})^2$$

15. (2) Let $z = x + iy$. Then,

$$\begin{aligned}x &= \lambda + 3 \text{ and } y = -\sqrt{5-\lambda^2} \\ \Rightarrow (x-3)^2 &= \lambda^2 \quad (1) \\ \text{and } y^2 &= 5 - \lambda^2 \quad (2)\end{aligned}$$

From Eqs. (1) and (2),

$$(x-3)^2 = 5 - y^2 \text{ or } (x-3)^2 + y^2 = 5$$

Obviously it is a semicircle as $y < 0$. Hence, part of the circle lies below the x -axis.

$$16. (1) \quad x + iy = 1 - t + i\sqrt{t^2 + t + 2}$$

$$\Rightarrow x = 1 - t, y = \sqrt{t^2 + t + 2}$$

Eliminating t ,

$$y^2 = t^2 + t + 2 = (1-x)^2 + 1 - x + 2 = \left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$$

$$\Rightarrow y^2 - \left(x - \frac{3}{2}\right)^2 = \frac{7}{4}, \text{ which is a hyperbola}$$

$$17. (4) \quad (x-3)^3 + 1 = 0$$

$$\Rightarrow \left(\frac{x-3}{-1}\right)^3 = 1$$

$$\Rightarrow \frac{x-3}{-1} = 1, \omega, \omega^2$$

$$\Rightarrow x = 2, 3 - \omega, 3 - \omega^2$$

Hence, the sum of complex roots is $6 - (\omega + \omega^2) = 6 + 1 = 7$.

$$18. (2) \quad x = \sqrt[3]{-1}$$

$$\Rightarrow x^3 = -1$$

$$\Rightarrow (-x)^3 = 1$$

$$\Rightarrow -x = 1, \omega, \omega^2$$

$$\Rightarrow x = -1, -\omega, -\omega^2$$

$$= -1, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$$

$$= -1, \frac{-\sqrt{3}+i}{2i}, \frac{\sqrt{3}+i}{2i}$$

$$= -1, \frac{-\sqrt{3}+\sqrt{-1}}{\sqrt{-4}}, \frac{\sqrt{3}+\sqrt{-1}}{\sqrt{-4}}$$

$$19. (4) \quad x^2 + x + 1 = 0$$

$$\Rightarrow x = \omega \text{ or } \omega^2$$

Let $x = \omega$. Then,

$$x + \frac{1}{x} = \omega + \frac{1}{\omega} = \omega + \omega^2 = -1$$

$$x^2 + \frac{1}{x^2} = \omega^2 + \frac{1}{\omega^2} = \omega^2 + \omega = -1$$

$$x^3 + \frac{1}{x^3} = \omega^3 + \frac{1}{\omega^3} = 2$$

$$x^4 + \frac{1}{x^4} = \omega^4 + \frac{1}{\omega^4} = \omega + \omega^2 = -1, \text{ etc.}$$

$$\begin{aligned}\therefore \left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 + \dots + \left(x^{27} + \frac{1}{x^{27}}\right)^2 \\ = \left[\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^4 + \frac{1}{x^4}\right)^2 + \dots + \left(x^{26} + \frac{1}{x^{26}}\right)^2\right] \\ + \left[\left(x^3 + \frac{1}{x^3}\right)^2 + \left(x^6 + \frac{1}{x^6}\right)^2 + \left(x^9 + \frac{1}{x^9}\right)^2 + \dots + \left(x^{27} + \frac{1}{x^{27}}\right)^2\right] \\ = 18 + 9(2^2) = 54\end{aligned}$$

$$20. (1) \quad \text{We have, } z^3 + 2z^2 + 2z + 1 = 0$$

$$\Rightarrow (z^3 + 1) + 2z(z + 1) = 0$$

$$\Rightarrow (z + 1)(z^2 + z + 1) = 0$$

$$\Rightarrow z = -1, \omega, \omega^2$$

Since $z = -1$ does not satisfy $z^{1985} + z^{100} + 1 = 0$ while $z = \omega, \omega^2$ satisfy it; hence, sum is $\omega + \omega^2 = -1$.

$$21. (4) \quad \text{Let } f(x) = 5x^3 + Mx + N.$$

$$\text{Also, } x^2 + x + 1 = (x - \omega)(x - \omega^2)$$

It is given that $f(x)$ is divisible by $x^2 + x + 1$.

$$\therefore f(\omega) = 5 + M\omega + N = 0$$

$$\text{and } f(\omega^2) = 5 + M\omega^2 + N = 0$$

So, $M = 0$ and $N = -5$.

$$\therefore M + N = -5$$

22. (1) Here $x = 4 \cos \theta, y = 4 \sin \theta$.

$$\begin{aligned}\therefore |x| - |y| &= |4 \cos \theta| - 4 |\sin \theta| \\ &= 4 |\cos \theta| - 4 |\sin \theta| \\ &= 4 \sqrt{1 - 2 |\cos \theta| |\sin \theta|} \\ &= 4 \sqrt{1 - |\sin 2\theta|}\end{aligned}$$

Hence, the range is $[0, 4]$.

23. (1) Let $z^3 = t$

$$\therefore t^2 - 6t + 25 = 0$$

$$\Rightarrow t = \frac{6 \pm \sqrt{-64}}{2} = 3 \pm 4i = z^3$$

$$\text{So, } |z^3| = 5$$

$$\Rightarrow |z| = 5^{1/3}$$

24. (1) $8iz^3 + 12z^2 - 18z + 27i = 0$

$$\text{or } 4iz^2(2z - 3i) - 9(2z - 3i) = 0$$

$$\text{or } (2z - 3i)(4iz^2 - 9) = 0$$

$$\Rightarrow z = \frac{3i}{2} \text{ and } z^2 = \frac{9}{4i}$$

$$\Rightarrow |z| = \frac{3}{2} \text{ and } |z^2| = \frac{9}{4}$$

$$\Rightarrow |z| = \frac{3}{2}$$

25. (1) Let $z_1 = a + ib$ and $z_2 = c - id$, where $a > 0$ and $d > 0$. Then,

$$|z_1| = |z_2| \Rightarrow a^2 + b^2 = c^2 + d^2 \quad (1)$$

$$\begin{aligned}\text{Now, } \frac{z_1 + z_2}{z_1 - z_2} &= \frac{(a+ib) + (c-id)}{(a+ib) - (c-id)} \\ &= \frac{[(a+c) + i(b-d)][(a-c) - i(b+d)]}{[(a-c) + i(b+d)][(a+c) - i(b-d)]} \\ &= \frac{(a^2 + b^2) - (c^2 + d^2) - 2(ad + bc)i}{a^2 + c^2 - 2ac + b^2 + d^2 + 2bd} \\ &= \frac{-(ad + bc)i}{a^2 + b^2 - ac + bd} \quad [\text{Using (1)}]\end{aligned}$$

Hence, $(z_1 + z_2)/(z_1 - z_2)$ is purely imaginary. However, if $ad + bc = 0$, then $(z_1 + z_2)/(z_1 - z_2)$ will be equal to zero. According to the conditions of the equation, we can have $ad + bc = 0$.

26. (1) We have

$$\arg\left(\frac{z_1}{z_2}\right) = \pi$$

$$\text{or } \arg(z_1) - \arg(z_2) = \pi$$

$$\text{or } \arg(z_1) = \arg(z_2) + \pi$$

$$\text{Let } \arg(z_2) = \theta. \text{ Then } \arg(z_1) = \pi + \theta.$$

$$\therefore z_1 = |z_1|[\cos(\pi + \theta) + i \sin(\pi + \theta)]$$

$$= |z_1|(-\cos \theta - i \sin \theta)$$

$$\text{and } z_2 = |z_2|(\cos \theta + i \sin \theta)$$

$$= |z_1|(\cos \theta + i \sin \theta) \quad (\because |z_1| = |z_2|)$$

$$= -z_1$$

$$\Rightarrow z_1 + z_2 = 0$$

27. (3) We have,

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$. Given,

$$\arg(z_1) - \arg(z_2) = 0$$

$$\Rightarrow |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$$

$$= (|z_1| - |z_2|)^2$$

$$\Rightarrow |z_1 - z_2| = ||z_1| - |z_2||$$

28. (2) Let $z_1 = |z_1|(\cos \theta_1 + i \sin \theta_1)$.

$$\text{Now, } \left|\frac{z_1}{z_2}\right| = 1 \Rightarrow |z_1| = |z_2|$$

$$\text{Also, } \arg(z_1 z_2) = 0 \text{ or } \arg(z_1) + \arg(z_2) = 0$$

$$\Rightarrow \arg(z_2) = -\theta_1$$

$$\begin{aligned}\Rightarrow z_2 &= |z_2|(\cos(-\theta_1) + i \sin(-\theta_1)) \\ &= |z_1|(\cos \theta_1 - i \sin \theta_1) = \bar{z}_1\end{aligned}$$

$$\Rightarrow \bar{z}_2 = \overline{(\bar{z}_1)} = z_1$$

$$\Rightarrow |z_2|^2 = z_1 z_2$$

29. (2) Let $A = x + iy$. Given,

$$|A| = 1 \Rightarrow x^2 + y^2 = 1$$

$$\text{and } |A + 1| = 1 \Rightarrow (x + 1)^2 + y^2 = 1$$

$$\Rightarrow x = -\frac{1}{2} \text{ and } y = \pm \frac{\sqrt{3}}{2}$$

$$\Rightarrow A = \omega \text{ or } \omega^2$$

$$\Rightarrow (\omega)^n = (1 + \omega)^n = (-\omega^2)^n$$

Therefore, n must be even and divisible by 3.

30. (3) Let $z = \cos x + i \sin x, x \in [0, 2\pi)$. Then,

$$1 = \left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right|$$

$$= \frac{|z^2 + \bar{z}^2|}{|z|^2}$$

$$= |\cos 2x + i \sin 2x + \cos 2x - i \sin 2x|$$

$$= 2 |\cos 2x|$$

$$\Rightarrow \cos 2x = \pm 1/2$$

$$\text{Now, } \cos 2x = 1/2$$

$$\Rightarrow x_1 = \frac{\pi}{6}, x_2 = \frac{5\pi}{6}, x_3 = \frac{7\pi}{6}, x_4 = \frac{11\pi}{6}$$

$$\cos 2x = -\frac{1}{2}$$

$$\Rightarrow x_5 = \frac{\pi}{3}, x_6 = \frac{2\pi}{3}, x_7 = \frac{4\pi}{3}, x_8 = \frac{5\pi}{3}$$

31. (3) $\bar{z} + i\bar{w} = 0$

$$\Rightarrow z - iw = 0$$

$$\Rightarrow z = iw$$

$$\arg zw = \pi$$

$$\text{or } \arg z + \arg w = \pi$$

$$\text{or } \arg z + \arg \frac{z}{i} = \pi$$

[Using (1)]

$$\text{or } \arg z + \arg z - \arg i = \pi$$

$$\text{or } 2 \arg z - \frac{\pi}{2} = \pi$$

$$\text{or } 2 \arg z = \frac{3\pi}{2}$$

$$\text{or } \arg z = \frac{3\pi}{4}$$

32. (3) $z = (3a - 7b) + i(3b + 7a)$

For z to be purely imaginary, we have

$$3a = 7b$$

$$\Rightarrow a = 7 \text{ or } b = 3 \text{ (for least value of } |z| \text{)}$$

$$\Rightarrow |z| = |3 + 7i|$$

33. (3) We have,

$$(\cos \theta + i \sin \theta) (\cos 2\theta + i \sin 2\theta) \cdots \times (\cos n\theta + i \sin n\theta) = 1$$

$$\text{or } \cos(\theta + 2\theta + 3\theta + \cdots + n\theta) + i \sin(\theta + 2\theta + \cdots + n\theta) = 1$$

$$\text{or } \cos\left(\frac{n(n+1)}{2}\theta\right) + i \sin\left(\frac{n(n+1)}{2}\theta\right) = 1$$

$$\text{or } \cos\left(\frac{n(n+1)}{2}\theta\right) = 1 \text{ and } \sin\left(\frac{n(n+1)}{2}\theta\right) = 0$$

$$\Rightarrow \frac{n(n+1)}{2}\theta = 2m\pi \Rightarrow \theta = \frac{4m\pi}{n(n+1)}, \text{ where } m \in \mathbb{Z}$$

34. (3) $z = (1 + i\sqrt{3})^{100} = 2^{100} \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)^{100}$

$$= 2^{100} \left(\cos \frac{100\pi}{3} + i \sin \frac{100\pi}{3} \right)$$

$$= 2^{100} \left(-\cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right)$$

$$= 2^{100} \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2} \right)$$

$$\Rightarrow \frac{\operatorname{Re}(z)}{\operatorname{Im}(z)} = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

35. (2) Let $\sin \frac{\pi}{8} + i \cos \frac{\pi}{8} = z$

$$\Rightarrow \left[\frac{1 + \sin \frac{\pi}{8} + i \cos \frac{\pi}{8}}{1 + \sin \frac{\pi}{8} - i \cos \frac{\pi}{8}} \right]^8$$

$$= \left(\frac{1 + z}{1 + \frac{1}{z}} \right)^8 = z^8$$

$$= \left(\sin \frac{\pi}{8} + i \cos \frac{\pi}{8} \right)^8$$

$$= \left(\cos \left(\frac{\pi}{2} - \frac{\pi}{8} \right) + i \sin \left(\frac{\pi}{2} - \frac{\pi}{8} \right) \right)^8$$

$$= \left(\cos \frac{3\pi}{8} + i \sin \frac{3\pi}{8} \right)^8 = \cos 3\pi = -1$$

36. (1) Let $z = x + iy$. Then,

$$|z - 3 - i| = |z - 9 - i|$$

$$\Rightarrow \sqrt{(x-3)^2 + (y-1)^2} = \sqrt{(x-9)^2 + (y-1)^2}$$

$$\Rightarrow x = 6$$

$$|z - 3 + 3i| = 3$$

$$\Rightarrow \sqrt{(x-3)^2 + (y+3)^2} = 3$$

$$\text{For } x = 6, y = -3.$$

$$\therefore z = 6 - 3i$$

37. (1) When $|z - 1| < |z + 1|$ (or $x > 0$)

$$|z| = |z - 1|$$

$$\Rightarrow x^2 + y^2 = (x-1)^2 + y^2$$

$$\Rightarrow x = 1/2$$

$$\Rightarrow z + \bar{z} = 1$$

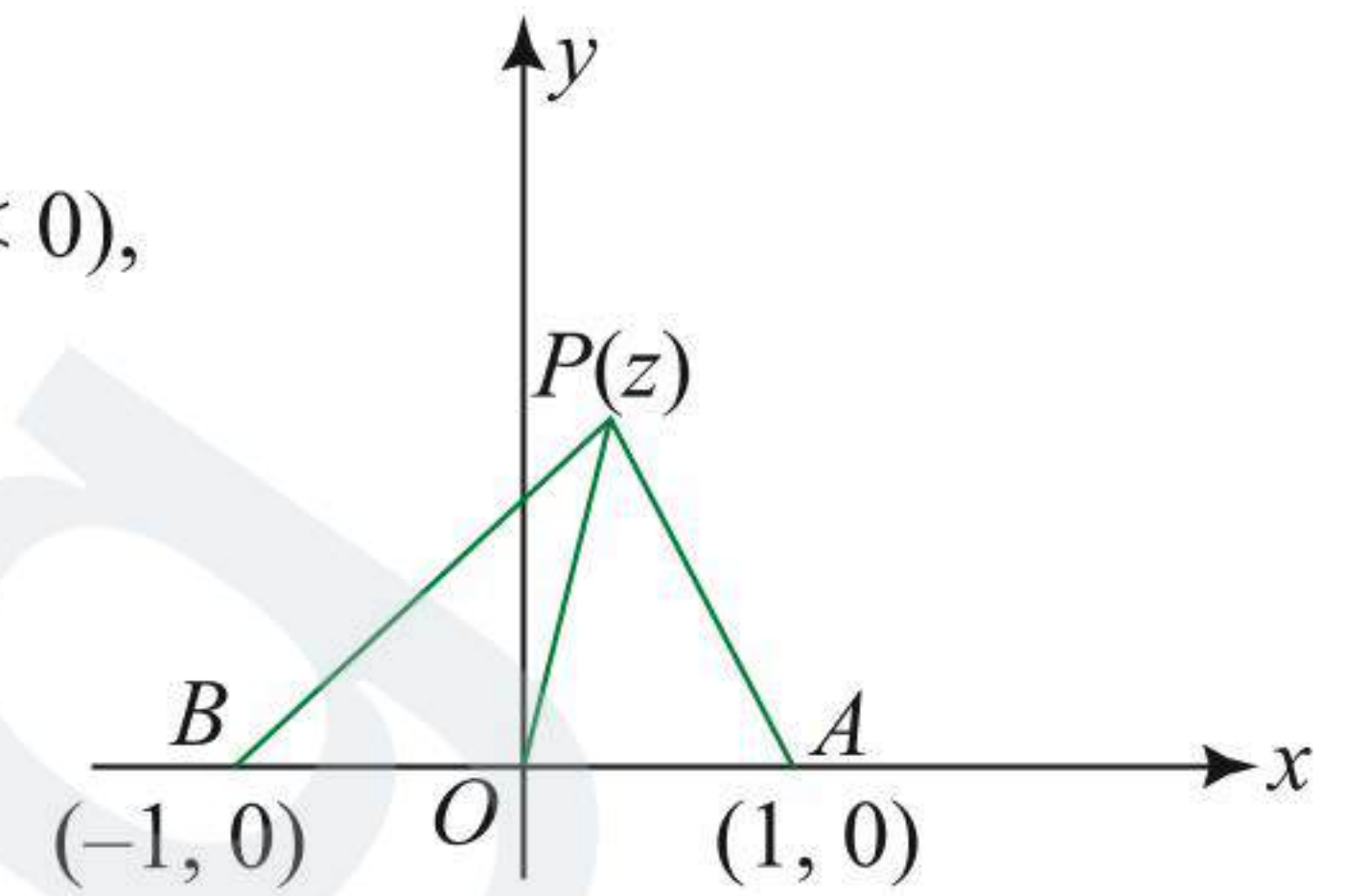
When $|z - 1| > |z + 1|$ (or $x < 0$),

$$|z| = |z + 1|$$

$$\Rightarrow x^2 + y^2 = (x+1)^2 + y^2$$

$$\Rightarrow x = -1/2$$

$$\Rightarrow z + \bar{z} = -1$$



38. (4) Let $z = x + iy$. Then,

$$|z^2 - 1| = |z|^2 + 1$$

$$\Rightarrow |(x^2 - y^2 - 1) + 2ixy| = x^2 + y^2 + 1$$

$$\Rightarrow (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2$$

$$\Rightarrow x = 0$$

Hence, z lies on imaginary axis.

39. (2) The given equation is

$$|z|^n = (z^2 + z) |z|^{n-2} + 1$$

$$\Rightarrow z^2 + z \text{ is real}$$

$$\Rightarrow z^2 + z = \bar{z}^2 + \bar{z}$$

$$\Rightarrow (z - \bar{z})(z + \bar{z} + 1) = 0$$

$$\Rightarrow z = \bar{z} = x \text{ as } z + \bar{z} + 1 \neq 0 \text{ (} x \neq -1/2 \text{)}$$

Hence, the given equation reduces to

$$x^n = x^n + x|x|^{n-2} + 1$$

$$\Rightarrow x|x|^{n-2} = -1$$

$$\Rightarrow x = -1$$

So number of solutions is 1.

40. (4) Given, $z^3 + \frac{3(\bar{z})^2}{|z|} = 0$

$$\text{Let } z = re^{i\theta}$$

$$\Rightarrow r^3 e^{i3\theta} + 3re^{-i2\theta} = 0$$

Since r cannot be zero, so

$$re^{i5\theta} = -3$$

which will hold for $r = 3$ and five distinct values of θ . Thus, there are five solutions.

41. (3) Let $z = a + ib$

$$\Rightarrow \bar{z} = a - ib$$

So, we have

$$z^{2008} = \bar{z}$$

$$\therefore |z|^{2008} = |\bar{z}| = |z|$$

$$\Rightarrow |z| [|z|^{2007} - 1] = 0$$

$$\Rightarrow |z| = 0 \text{ or } |z| = 1$$

If $|z| = 0$, then $z = 0$.

If $|z| = 1$, then $z^{2009} = z\bar{z} = |z|^2 = 1$.

So, there are 2009 values of z .

Therefore, total number of ordered pairs is 2010.

42. (2) We have $az^3 + bz^2 + \bar{b}z + \bar{a} = 0$... (1)

Taking conjugate of both sides, we get

$$\Rightarrow \bar{a}\bar{z}^3 + \bar{b}\bar{z}^2 + b\bar{z} + a = 0$$

Dividing this equation by $\bar{z}^3$ and writing the terms in reverse order, we get

$$\frac{a}{\bar{z}^3} + \frac{b}{\bar{z}^2} + \frac{\bar{b}}{\bar{z}} + \bar{a} = 0 \quad [\text{For } \bar{z} \neq 0] \quad \dots (2)$$

Since equations (1) and (2) are identical,

$$z^3 = \frac{1}{\bar{z}^3} \Rightarrow |z|^6 = 1 \Rightarrow |z| = 1$$

Therefore, if α is a root of the given equation, then $|\alpha| = 1$.

$$43. (1) \alpha = \frac{z - \bar{w}}{k^2 + z\bar{w}} \Rightarrow \bar{\alpha} = \frac{\bar{z} - w}{k^2 + \bar{z}w}$$

But $z\bar{z} = w\bar{w} = k^2$. Hence,

$$\Rightarrow \bar{\alpha} = \frac{\frac{k^2}{z} - \frac{k^2}{\bar{w}}}{k^2 + \frac{k^2}{z} \frac{k^2}{\bar{w}}} = \frac{\bar{w} - z}{z\bar{w} + k^2} = -\alpha$$

$$\Rightarrow \alpha + \bar{\alpha} = 0$$

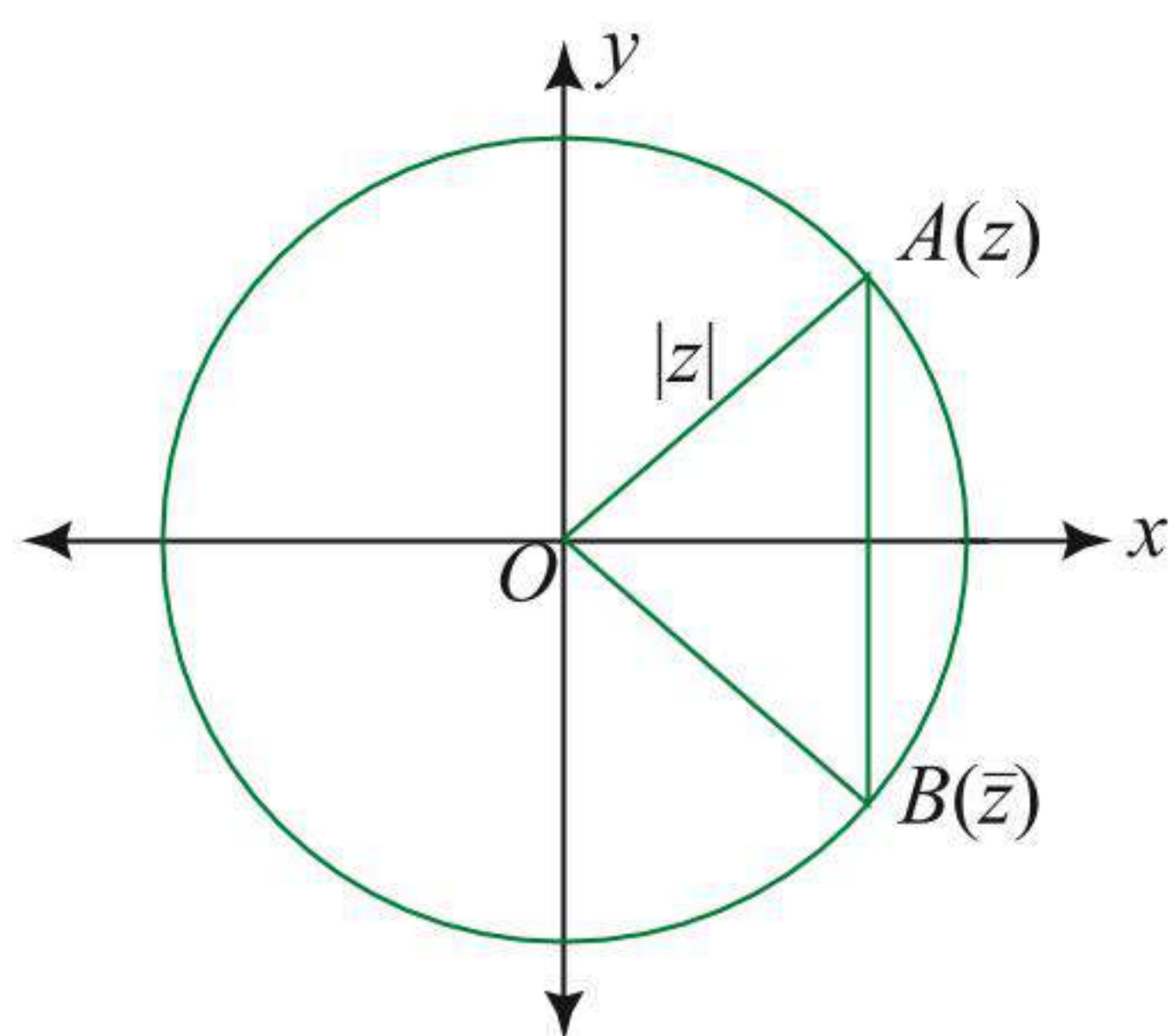
$$\Rightarrow \operatorname{Re}(\alpha) = 0$$

$$44. (1) \text{ Assuming } \arg z_1 = \theta \text{ and } \arg z_2 = \theta + \alpha,$$

$$\begin{aligned} \frac{az_1}{bz_2} + \frac{bz_2}{az_1} &= \frac{a|z_1|e^{i\theta}}{b|z_2|e^{i(\theta+\alpha)}} + \frac{b|z_2|e^{i(\theta+\alpha)}}{a|z_1|e^{i\theta}} \\ &= e^{-i\alpha} + e^{i\alpha} = 2 \cos \alpha \end{aligned}$$

Hence, the point lies on the line segment $[-2, 2]$ of the real axis.

$$45. (1)$$



$$|z - \bar{z}| = AB$$

$$\text{while } |z|(\arg z - \arg \bar{z}) = \text{Arc } AB$$

$$\therefore |z - \bar{z}| \leq |z|(\arg z - \arg \bar{z})$$

$$46. (3) \text{ Let } a = \cos \alpha + i \sin \alpha$$

$$b = \cos \beta + i \sin \beta$$

$$c = \cos \gamma + i \sin \gamma$$

$$\text{Then, } a + 2b + 3c = (\cos \alpha + 2 \cos \beta + 3 \cos \gamma) + i(\sin \alpha + 2 \sin \beta + 3 \sin \gamma) = 0$$

$$\Rightarrow a^3 + 8b^3 + 27c^3 = 18abc$$

$$\Rightarrow \cos 3\alpha + 8 \cos 3\beta + 27 \cos 3\gamma = 18 \cos(\alpha + \beta + \gamma)$$

$$\text{and } \sin 3\alpha + 8 \sin 3\beta + 27 \sin 3\gamma = 18 \sin(\alpha + \beta + \gamma)$$

$$47. (1) u^2 - 2u + 2 = 0 \Rightarrow u = 1 \pm i$$

$$\begin{aligned} \Rightarrow & \frac{(x + \alpha)^n - (x + \beta)^n}{\alpha - \beta} \\ &= \frac{[(\cot \theta - 1) + (1 + i)]^n - [(\cot \theta - 1) + (1 - i)]^n}{2i} \\ & \quad (\because \cot \theta - 1 = x) \\ &= \frac{(\cos \theta + i \sin \theta)^n - (\cos \theta - i \sin \theta)^n}{\sin^n \theta \cdot 2i} \\ &= \frac{2i \sin n\theta}{\sin^n \theta \cdot 2i} \\ &= \frac{\sin n\theta}{\sin^n \theta} \end{aligned}$$

$$48. (1) i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = e^{i\frac{\pi}{2}}$$

$$\Rightarrow i^i = \left(e^{i\frac{\pi}{2}} \right)^i = e^{-\frac{\pi}{2}}$$

$$\Rightarrow z = (i)^{i^i} = i^{e^{-\frac{\pi}{2}}}$$

$$\Rightarrow |z| = 1$$

$$49. (4) z = i \log(2 - \sqrt{3})$$

$$\Rightarrow e^{iz} = e^{i^2 \log(2 - \sqrt{3})} = e^{-\log(2 - \sqrt{3})}$$

$$= e^{\log(2 - \sqrt{3})^{-1}} = e^{\log(2 + \sqrt{3})} = (2 + \sqrt{3})$$

$$\Rightarrow \cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{(2 + \sqrt{3}) + (2 - \sqrt{3})}{2} = 2$$

$$50. (1) |z| = 1, \text{ let } \alpha = -1 + 3z$$

$$\Rightarrow \alpha + 1 = 3z$$

$$\Rightarrow |\alpha + 1| = 3|z| = 3$$

Hence, α lies on a circle centered at -1 and radius equal to 3.

$$51. (2) \text{ Let } z = x + iy. \text{ Then,}$$

$$\operatorname{Re}\left(\frac{1}{z}\right) = k$$

$$\Rightarrow \operatorname{Re}\left(\frac{1}{x + iy}\right) = k$$

$$\Rightarrow \operatorname{Re}\left(\frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}\right) = k$$

$$\Rightarrow \frac{x}{x^2 + y^2} = k$$

$$\Rightarrow x^2 + y^2 - \frac{1}{k}x = 0$$

which is a circle.

$$52. (2) 2\left|z - \frac{1}{2}\right| = |z - 1|$$

$$\therefore \frac{|z - 1|}{\left|z - \frac{1}{2}\right|} = 2$$

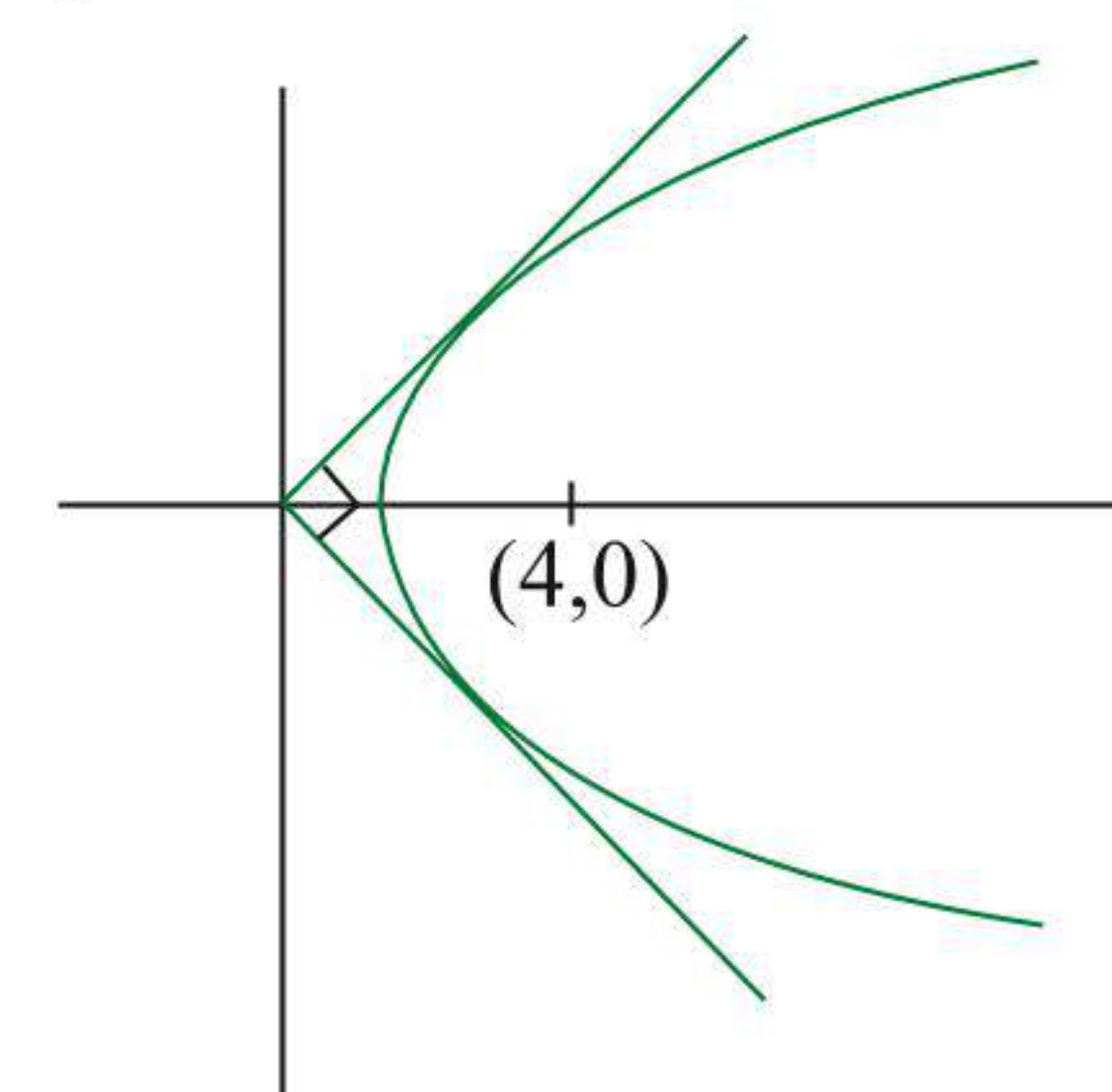
So, locus of z is a circle.

$$53. (4) |z - 4| = \operatorname{Re}(z)$$

$$\Rightarrow \sqrt{(x - 4)^2 + y^2} = x$$

$$\text{or } x^2 - 8x + 16 + y^2 = x^2$$

$$\text{or } y^2 = 8(x - 2)$$



The given relation represents the parabola with focus $(4, 0)$ and the imaginary axis as the directrix. The two tangents from directrix are at right angle. Hence, greatest positive argument of z is $\pi/4$.

$$54. (3) \quad z = \frac{at+b}{t-c} \Rightarrow t = \frac{b+c}{z-a}$$

Now, $|t| = 1$

$$\Rightarrow \left| \frac{b+c}{z-a} \right| = 1$$

$$\Rightarrow \left| \frac{z+\frac{b}{c}}{z-a} \right| = \frac{1}{|c|} \quad (\neq 1 \text{ as } |c| \neq |t|)$$

$\Rightarrow$ locus of z is a circle

$$55. (3) \quad z^2 + z|z| + |z|^2 = 0$$

$$\Rightarrow \left(\frac{z}{|z|} \right)^2 + \frac{z}{|z|} + 1 = 0$$

$$\Rightarrow \frac{z}{|z|} = \omega, \omega^2$$

$$\Rightarrow z = \omega|z| \text{ or } z = \omega^2|z|$$

$$\Rightarrow x+iy = |z| \left(\frac{-1}{2} + \frac{i\sqrt{3}}{2} \right) \text{ or } x+iy = |z| \left(\frac{-1}{2} - \frac{i\sqrt{3}}{2} \right)$$

$$\Rightarrow x = -\frac{1}{2}|z|, y = |z| \frac{\sqrt{3}}{2} \text{ or } x = -\frac{1}{2}|z|, y = -|z| \frac{\sqrt{3}}{2}$$

$$\Rightarrow y + \sqrt{3}x = 0 \text{ or } y - \sqrt{3}x = 0 \Rightarrow y^2 - 3x^2 = 0$$

$$56. (1) \quad \log_{1/3} \left(\frac{|z-3|^2 + 2}{11|z-3|-2} \right) > 1, 11|z-3|-2 > 0$$

$$\Rightarrow \frac{|z-3|^2 + 2}{11|z-3|-2} < \frac{1}{3}$$

$$\Rightarrow (3t-8)(t-1) < 0 \text{ (where } |z-3| = t)$$

$$\Rightarrow 1 < |z-3| < 8/3$$

Hence, z lies between the two concentric circles.

57. (3) We have,

$$|(x-2) + i(y-1)| = |z| \left| \frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta \right|$$

where $\theta = \arg z$.

$$\sqrt{(x-2)^2 + (y-1)^2} = \frac{1}{\sqrt{2}} |x-y|,$$

which represents a parabola

$$58. (4) \quad |\omega z - 1 - \omega^2| = a$$

$$\Rightarrow |z+1| = a \Rightarrow |z-1+2| = a$$

$$\Rightarrow |z-1| + 2 \geq a \Rightarrow 0 \leq a \leq 4$$

$$59. (2) \quad |z^2 - 3| \geq |z|^2 - 3$$

$$\Rightarrow 3|z| \geq |z|^2 - 3$$

$$\Rightarrow |z|^2 - 3|z| - 3 \leq 0$$

$$\Rightarrow 0 < |z| \leq \frac{3+\sqrt{21}}{2}$$

$$60. (2) \quad |2z-1| = |z-2|$$

$$\text{or } |2z-1|^2 = |z-2|^2$$

$$\text{or } (2z-1)(2\bar{z}-1) = (z-2)(\bar{z}-2)$$

$$\text{or } 4z\bar{z} - 2\bar{z} - 2z + 1 = z\bar{z} - 2\bar{z} - 2z + 4$$

$$\text{or } 3|z|^2 = 3$$

$$\text{or } |z| = 1$$

Again,

$$\begin{aligned} |z_1 + z_2| &= |z_1 - \alpha + z_2 - \beta + \alpha + \beta| \\ &\leq |z_1 - \alpha| + |z_2 - \beta| + |\alpha + \beta| \\ &< \alpha + \beta + |\alpha + \beta| \\ &= 2|\alpha + \beta| \end{aligned} \quad [\because \alpha, \beta > 0]$$

$$\therefore \left| \frac{z_1 + z_2}{\alpha + \beta} \right| < 2$$

$$\text{or } \left| \frac{z_1 + z_2}{\alpha + \beta} \right| < 2|z|$$

$$61. (1) \quad a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_{n-1} z + a_n = 3$$

$$\text{or } |3| = |a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n|$$

$$\text{or } 3 \leq |a_0| |z|^n + |a_1| |z|^{n-1} + \dots + |a_{n-1}| |z| + |a_n|$$

$$\text{or } 3 < 2(|z|^n + |z|^{n-1} + \dots + |z| + 1) \quad (\because |a_i| < 2)$$

$$\text{or } 1 + |z| + |z|^2 + \dots + |z|^n > \frac{3}{2}$$

If $|z| \geq 1$, the inequality is clearly satisfied. For $|z| < 1$, we have,

$$\frac{1 - |z|^{n+1}}{1 - |z|} > \frac{3}{2}$$

$$\text{or } 2 - 2|z|^{n+1} > 3 - 3|z|$$

$$\text{or } 2|z|^{n+1} < 3|z| - 1$$

$$\text{or } 3|z| - 1 > 0$$

$$\text{or } |z| > \frac{1}{3}$$

$$62. (1) \quad |z^2 + 2z \cos \alpha| \leq |z|^2 + |2z \cos \alpha|$$

$$= |z|^2 + 2|z| |\cos \alpha|$$

$$\leq |z|^2 + 2|z|$$

$$< (\sqrt{2}-1)^2 + 2(\sqrt{2}-1) = 1$$

$$(\because |\cos \alpha| \leq 1)$$

$$63. (3) \quad \left| \sum_{r=1}^n z_r \right| \leq \sum_{r=1}^n |z_r| \leq \sum_{r=1}^n |z_r - r| + \sum_{r=1}^n r \leq 2 \sum_{r=1}^n r$$

$$64. (2) \quad 11z^{10} + 10iz^9 + 10iz - 11 = 0$$

$$\text{or } z^9 (11z + 10i) = 11 - 10iz$$

$$\text{or } z^9 = \frac{11 - 10iz}{11z + 10i}$$

$$\text{or } |z^9| = \frac{|11 - 10iz|}{|11z + 10i|}$$

$$\text{Now } |11 - 10iz|^2 - |11z + 10i|^2 = 21(1 - |z|^2)$$

For $|z| < 1$

$$|11 - 10iz|^2 - |11z + 10i|^2 > 0$$

$$\Rightarrow |z^9| = \frac{|11 - 10iz|}{|11z + 10i|} > 1$$

i.e. $|z^9| > 1$ which contradicts with $|z| < 1$

For $|z| > 1$ we get $|z^9| < 1$

$$\Rightarrow |z| = 1$$

$$65. (2) \quad 2z^2 + 2z + \lambda = 0$$

Let the roots be z_1, z_2 . Then,

$$z_1 + z_2 = -1 \text{ and } z_1 z_2 = \frac{\lambda}{2}$$

0, z_1, z_2 form an equilateral triangle.

$$\therefore z_1^2 + z_2^2 = z_1 z_2$$

$$\Rightarrow (z_1 + z_2)^2 = 3z_1 z_2$$

$$\Rightarrow 1 = 3 \frac{\lambda}{2}$$

$$\text{or } \lambda = \frac{2}{3}$$

$$66. (3) S_1 = \Sigma z_1 = -3a, S_2 = \Sigma z_1 z_2 = 3b$$

Since the triangle is equilateral, we have

$$\Sigma z_1^2 = \Sigma z_1 z_2$$

$$\Rightarrow (\Sigma z_1)^2 - 2 \Sigma z_1 z_2 = \Sigma z_1 z_2$$

$$\Rightarrow (\Sigma z_1)^2 = 3 \Sigma z_1 z_2$$

$$\Rightarrow (-3a)^2 = 3(3b)$$

$$\Rightarrow 9a^2 = 9b$$

$$\Rightarrow a^2 = b$$

$$67. (2) \text{ Taking cube roots of both sides, we get}$$

$$z + ab = a(1)^{1/3} = a, a\omega, a\omega^2$$

$$\text{where } \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\therefore z_1 = a - ab, z_2 = a\omega - ab, z_3 = a\omega^2 - ab$$

$$|z_1 - z_2| = |a(1 - \omega)|$$

$$= |a| \left| 1 - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) \right|$$

$$= |a| \left| \frac{3}{2} - i\frac{\sqrt{3}}{2} \right|$$

$$= |a| \left(\frac{9}{4} + \frac{3}{4} \right)^{1/2} = \sqrt{3} |a|$$

$$\text{Similarly, } |z_2 - z_3| = |z_3 - z_1| = \sqrt{3} |a|$$

$$68. (1) |z_1| = |z_2| = |z_3| = 1$$

Hence, the circumcenter of triangle is origin. Also, centroid $(z_1 + z_2 + z_3)/3 = 0$, which coincides with the circumcenter. So the triangle is equilateral. Since radius is 1, length of side is $a = \sqrt{3}$. Therefore, the area of the triangle is $(\sqrt{3}/4)a^2 = (3\sqrt{3}/4)$.

$$69. (3) \text{ We have } 2 = |z + i\omega| \leq |z| + |\omega| \quad (\because |z_1 + z_2| \leq |z_1| + |z_2|)$$

$$\therefore |z| + |\omega| \leq 2 \quad \dots(1)$$

But it is given that $|z| \leq 1$ and $|\omega| \leq 1$.

$$\Rightarrow |z| + |\omega| \leq 2 \quad \dots(2)$$

From (1) and (2),

$$|z| = |\omega| = 1$$

$$\text{Also, } |z + i\omega| = |z - i\bar{\omega}|$$

$$\Rightarrow |z - (-i\omega)| = |z - (i\bar{\omega})|$$

This means that z lies on perpendicular bisector of the line segment joining $(-i\omega)$ and $(i\bar{\omega})$, which is real axis, as $(-i\omega)$ and $(i\bar{\omega})$ are conjugate to each other.

For z , $\text{Im}(z) = 0$

If $z = x$, then $|z| \leq 1$

$$\Rightarrow x^2 \leq 1$$

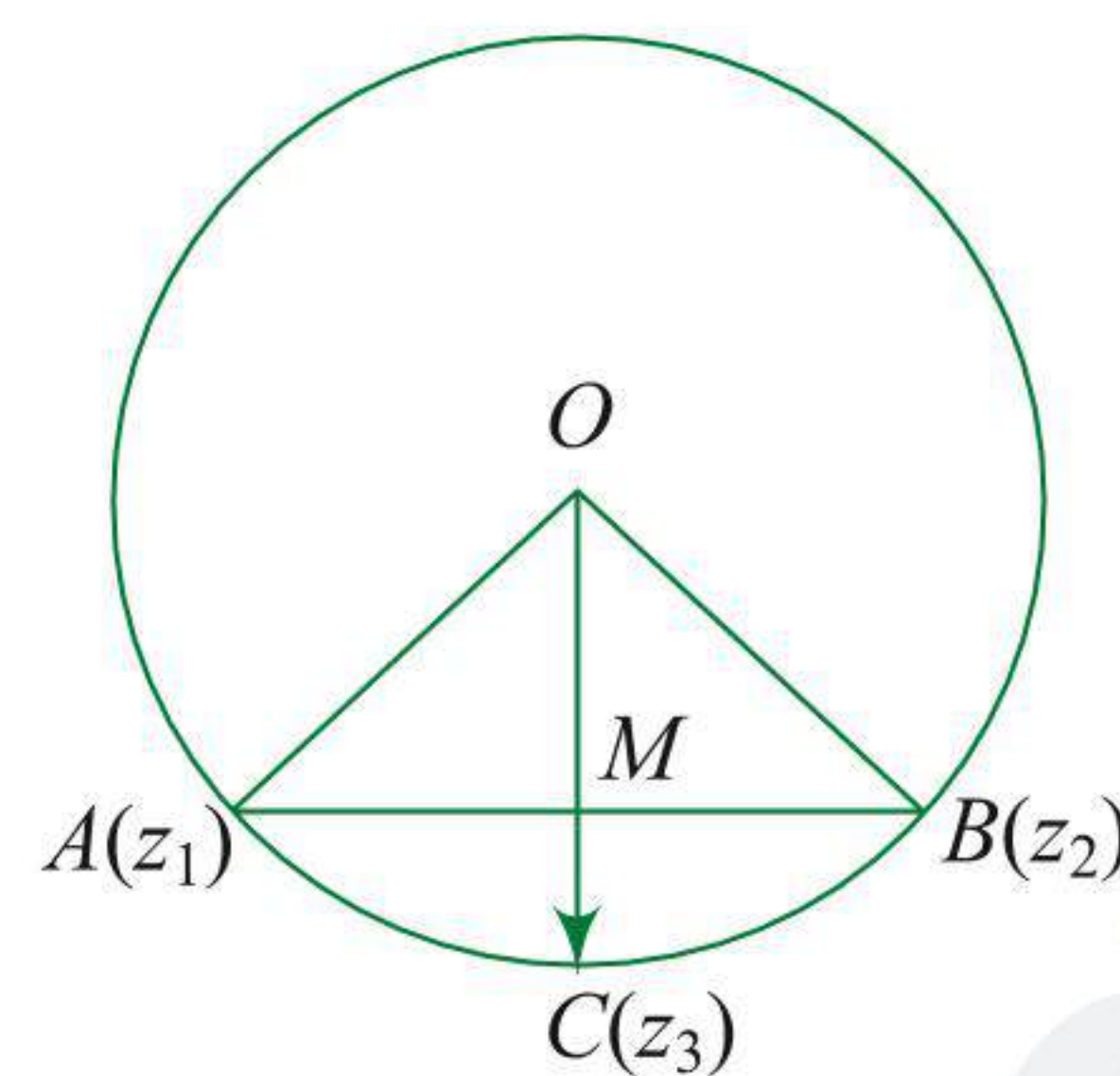
$$\Rightarrow -1 \leq x \leq 1$$

Therefore, (3) is the correct option.

$$70. (4) \text{ Given that } 4z_3 = 3(z_1 + z_2)$$

$$\therefore \frac{z_1 + z_2}{2} = \frac{2}{3} z_3$$

Therefore, mid-point of AB is the point which divides OM in the ratio 2 : 1.



$$\text{or } OM = \frac{2}{3} OC = \frac{2}{3} |z_3| = \frac{2}{3}$$

$$\therefore |z_1 - z_2| = AB = 2AM = 2\sqrt{1 - \frac{4}{9}} = \frac{2\sqrt{5}}{3}$$

$$71. (1)$$



The first condition implies that $(z_1 + z_3)/2 = (z_2 + z_4)/2$, i.e., diagonals AC and BD bisect each other. Hence, quadrilateral is a parallelogram. The second condition implies that the angle between AD and AB is 90° . Hence, the parallelogram is a rectangle.

$$72. (3) \text{ Given that } |k + z^2| = |z^2| - k = |z^2| + |k|$$

$$\Rightarrow k, z^2 \text{ and } 0 + i0 \text{ are collinear}$$

$$\Rightarrow \arg(z^2) = \arg(k)$$

$$\Rightarrow 2 \arg(z) = \pi$$

$$\Rightarrow \arg(z) = \frac{\pi}{2}$$

$$73. (3) |z_1 - i| = |z_2 - i| = |z_3 - i|$$

Hence, z_1, z_2, z_3 lie on the circle whose center is i . Also given that the triangle is equilateral. Hence, centroid and circumcenter coincides.

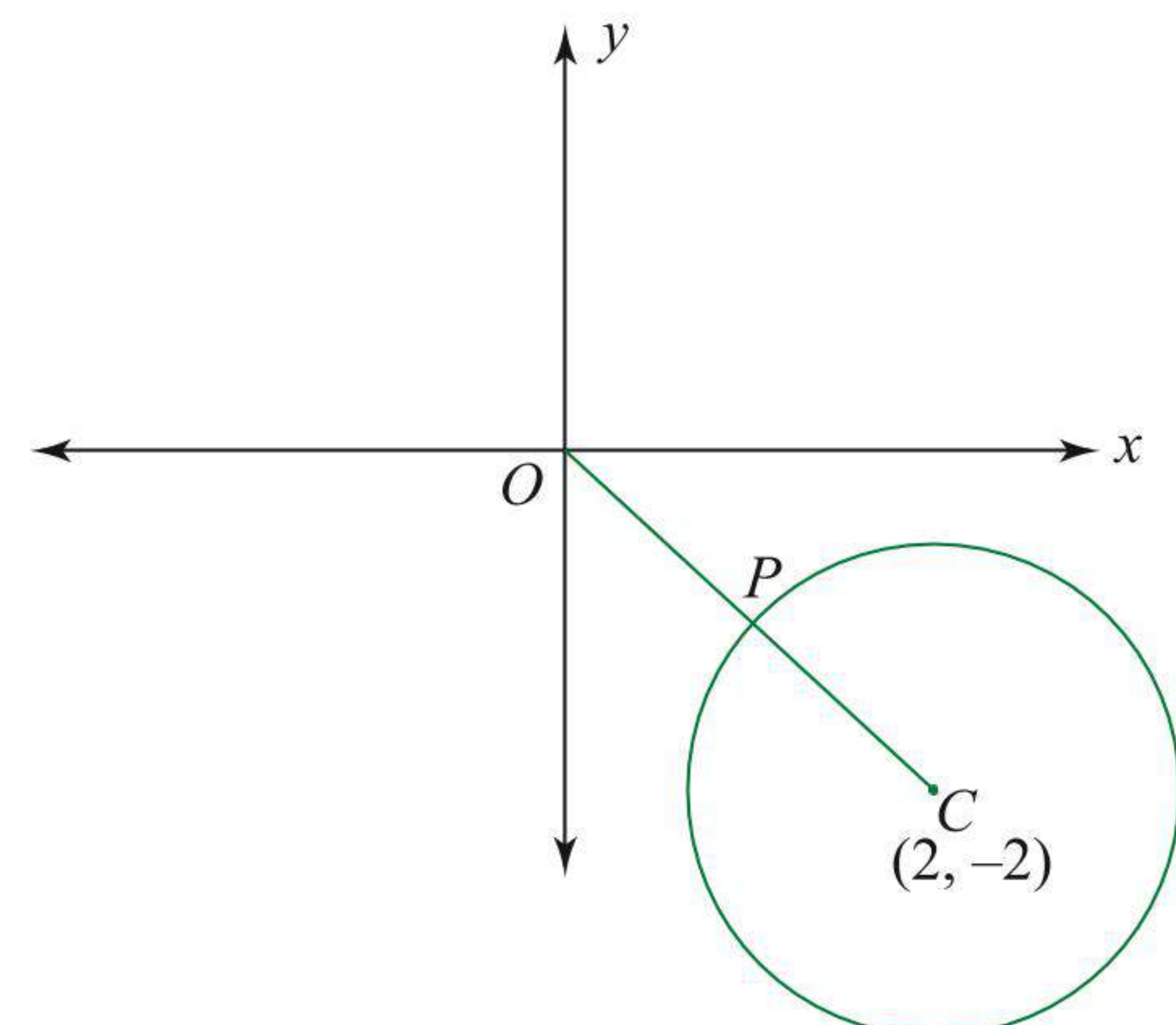
$$\therefore \frac{z_1 + z_2 + z_3}{3} = i$$

$$\Rightarrow |z_1 + z_2 + z_3| = 3$$

$$74. (1) \text{ We have,}$$

$$|z - 2 + 2i| = 1$$

$$\Rightarrow |z - (2 - 2i)| = 1$$



Hence, z lies on a circle having center at $(2, -2)$ and radius 1. It is evident from the figure that the required complex number z is given by the point P .

We find that OP makes an angle $\pi/4$ with OX and

$$\begin{aligned} OP &= OC - CP \\ &= \sqrt{2^2 + 2^2} - 1 = 2\sqrt{2} - 1 \end{aligned}$$

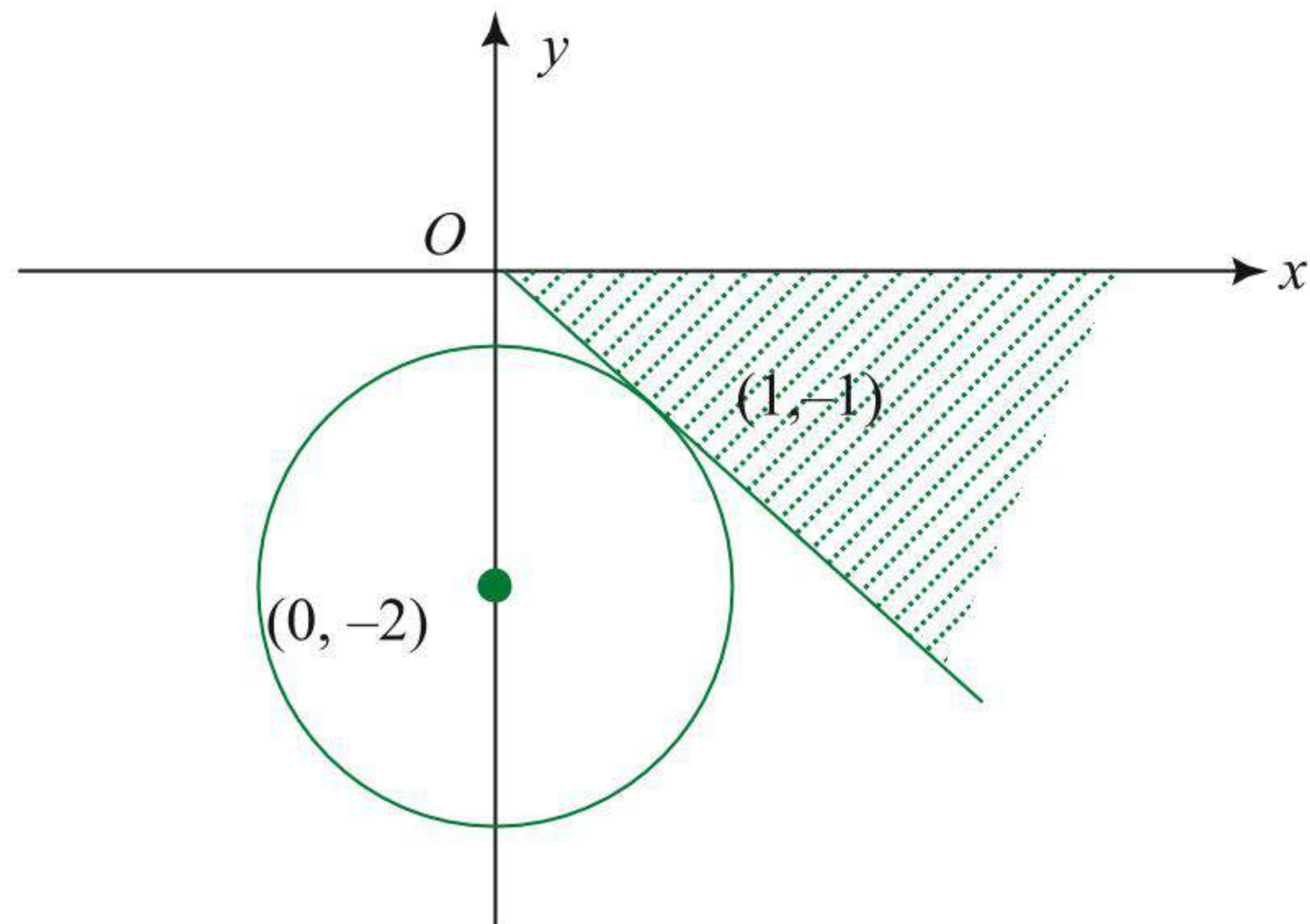
So, coordinates of P are $[(2\sqrt{2} - 1)\cos(\pi/4),$

$$-(2\sqrt{2} - 1)\sin(\pi/4)], \text{ i.e., } ((2 - 1/\sqrt{2}), -(2 - 1/\sqrt{2})).$$

Hence,

$$z = \left(2 - \frac{1}{\sqrt{2}}\right) + \left\{-\left(2 - \frac{1}{\sqrt{2}}\right)\right\}i = \left(2 - \frac{1}{\sqrt{2}}\right)(1 - i)$$

75. (3) $kz/(k+1)$ represents any point lying on the line joining origin and z .

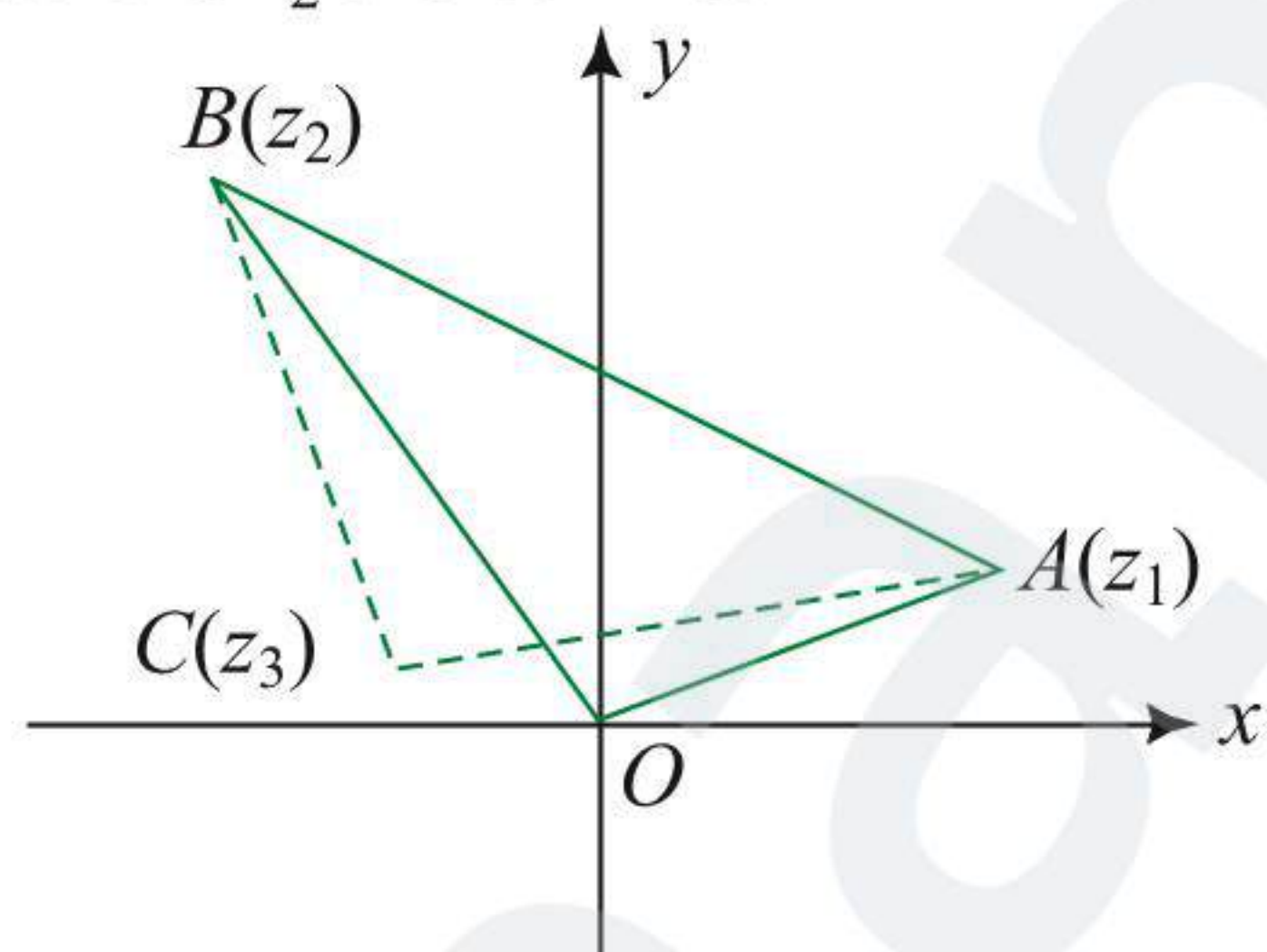


Given, $\left|\frac{kz}{k+1} + 2i\right| > \sqrt{2}$

Hence, $kz/(k+1)$ should lie outside the circle $|z + 2i| > \sqrt{2}$. So, z should lie in the shaded region.

$$\therefore -\frac{\pi}{4} < \arg(z) < 0$$

76. (4) $|z_2 + iz_1| = |z_1| + |z_2| = |iz_1| + |z_2|$
 $\Rightarrow iz_1, 0 + i0$ and z_2 are collinear



$$\Rightarrow \arg(iz_1) = \arg(z_2)$$

$$\Rightarrow \arg(z_2) - \arg(z_1) = \frac{\pi}{2}$$

Let $z_3 = \frac{z_2 - iz_1}{1 - i}$

or $(1 - i)z_3 = z_2 - iz_1$

or $z_2 - z_3 = i(z_1 - z_3)$

$$\therefore \angle ACB = \frac{\pi}{2}$$

and $|z_1 - z_3| = |z_2 - z_3|$

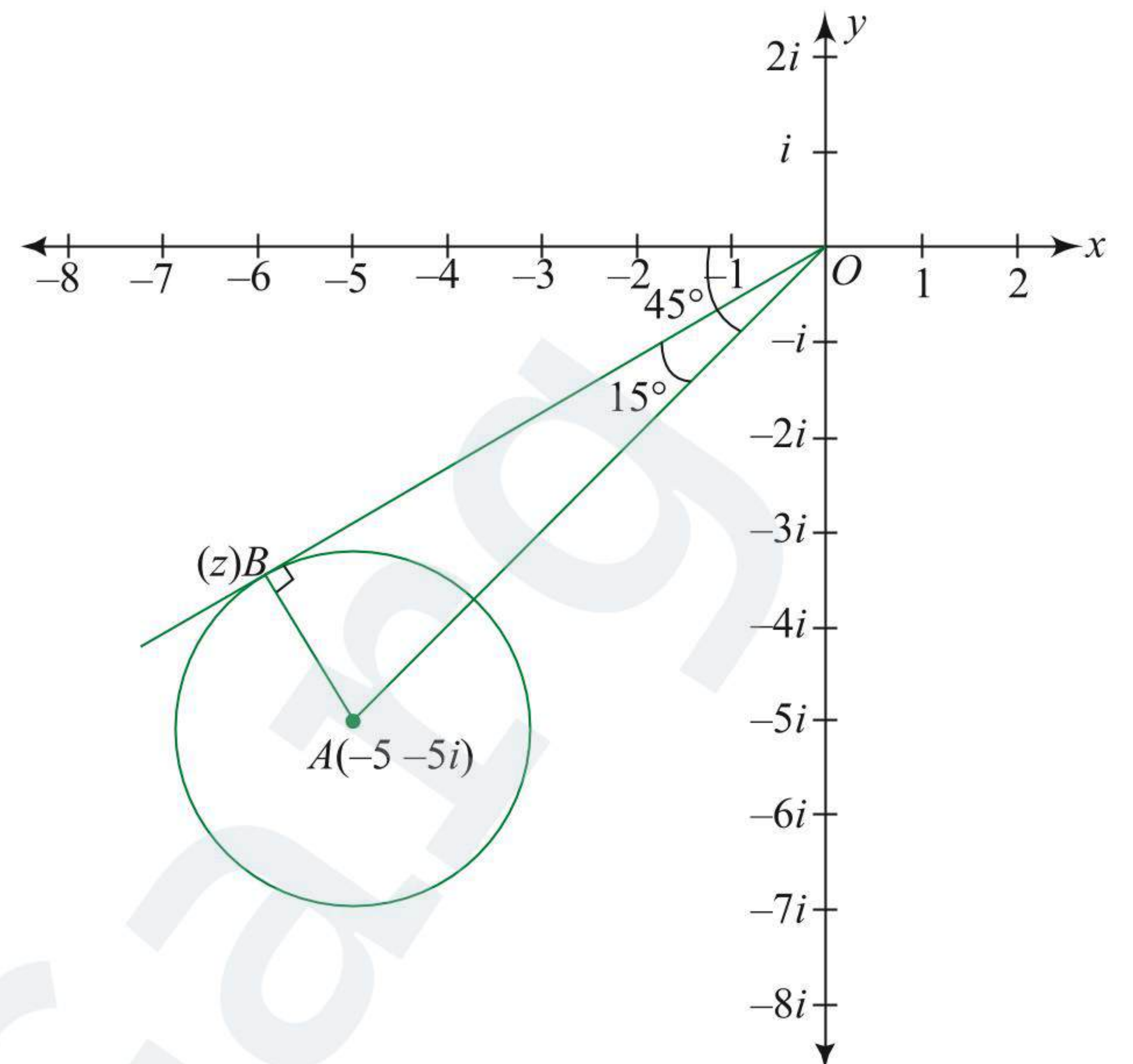
$$\Rightarrow AC = BC$$

$$\therefore AB^2 = AC^2 + BC^2$$

$$\Rightarrow AC = \frac{5}{\sqrt{2}} \quad (\because AB = 5)$$

Therefore, area of ΔABC is $(1/2)AC \times BC = AC^2/2 = 25/4$ sq. units

77. (1)



$$|2z + 10 + 10i| \leq (5\sqrt{3} - 5)$$

$$\Rightarrow |z + 5 + 5i| \leq \frac{5\sqrt{3} - 5}{2}$$

Point B has the least principal argument.

Now, $AB = \frac{5(\sqrt{3} - 1)}{2}$

$$OA = 5\sqrt{2}$$

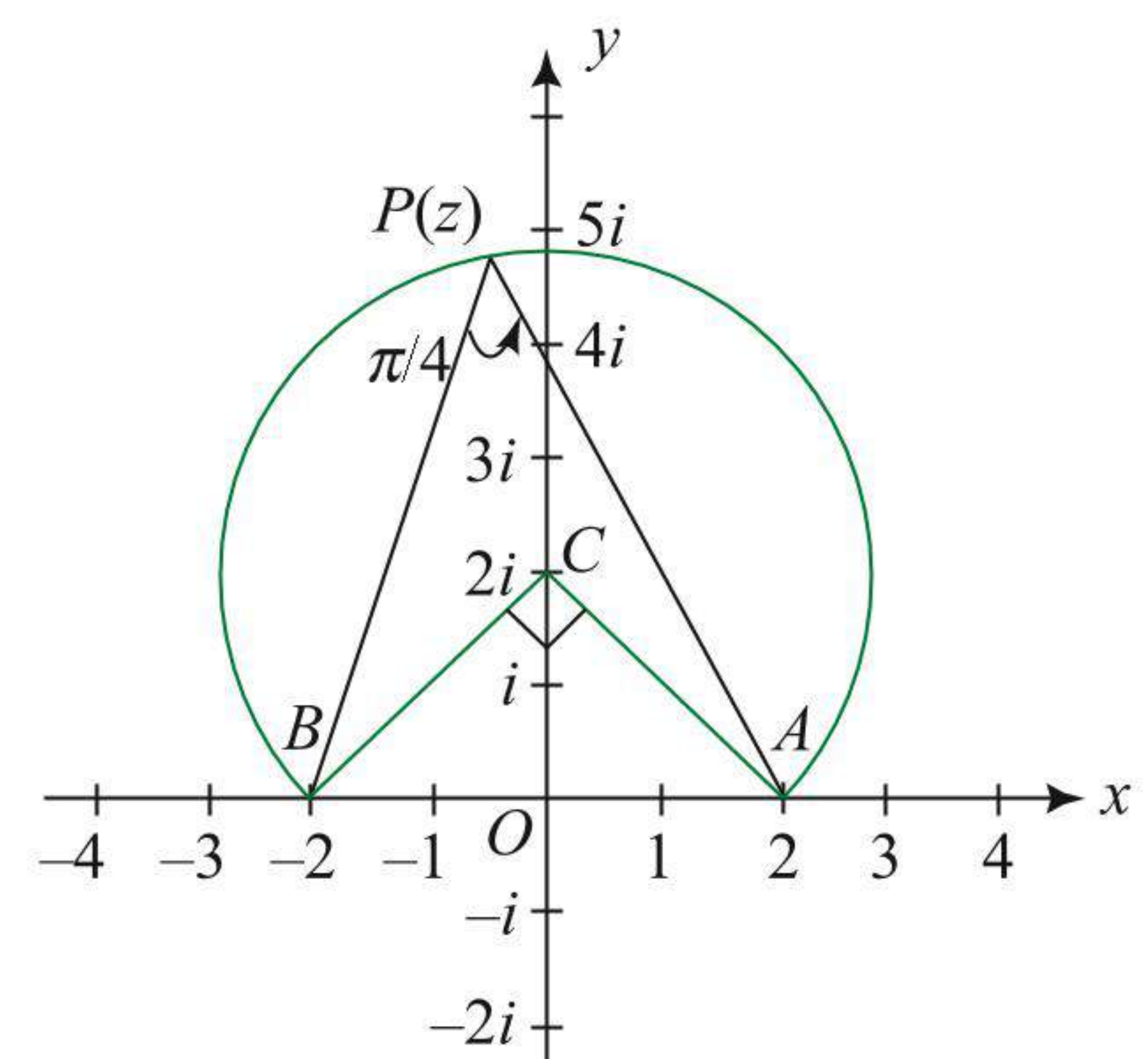
$$\therefore \angle AOB = \frac{\pi}{12}$$

$$\therefore \arg(z) = -\frac{5\pi}{6}$$

78. (2) $CA = CB = 2\sqrt{2}, OC = 2$

$$\Rightarrow OA = OB = 2$$

$$\Rightarrow A \equiv 2 + 0i, B \equiv -2 + 0i$$



Clearly,

$$\angle BCA = \pi/2$$

$$\Rightarrow \angle BPA = \pi/4$$

$$\Rightarrow \arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}$$

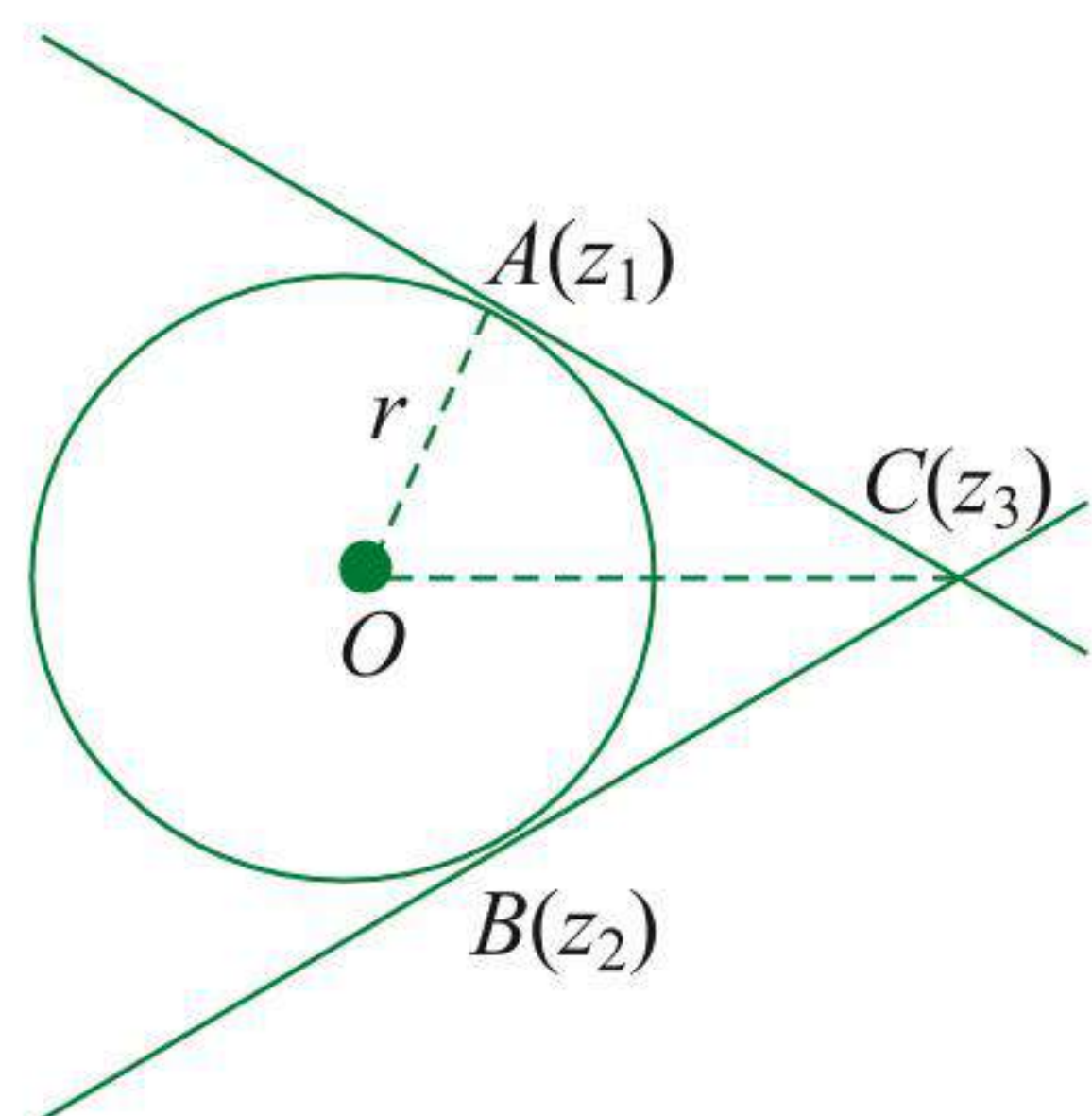
79. (2) As $\triangle OAC$ is a right-angled triangle with right angle at A , so

$$\begin{aligned} |z_1|^2 + |z_3 - z_1|^2 &= |z_3|^2 \\ \Rightarrow 2|z_1|^2 - \bar{z}_3 z_1 - \bar{z}_1 z_3 &= 0 \\ \Rightarrow 2\bar{z}_1 - \bar{z}_3 - \frac{\bar{z}_1}{z_1} z_3 &= 0 \end{aligned} \quad (1)$$

Similarly,

$$2\bar{z}_2 - \bar{z}_3 - \frac{\bar{z}_2}{z_2} z_3 = 0 \quad (2)$$

Subtracting (2) from (1), we get

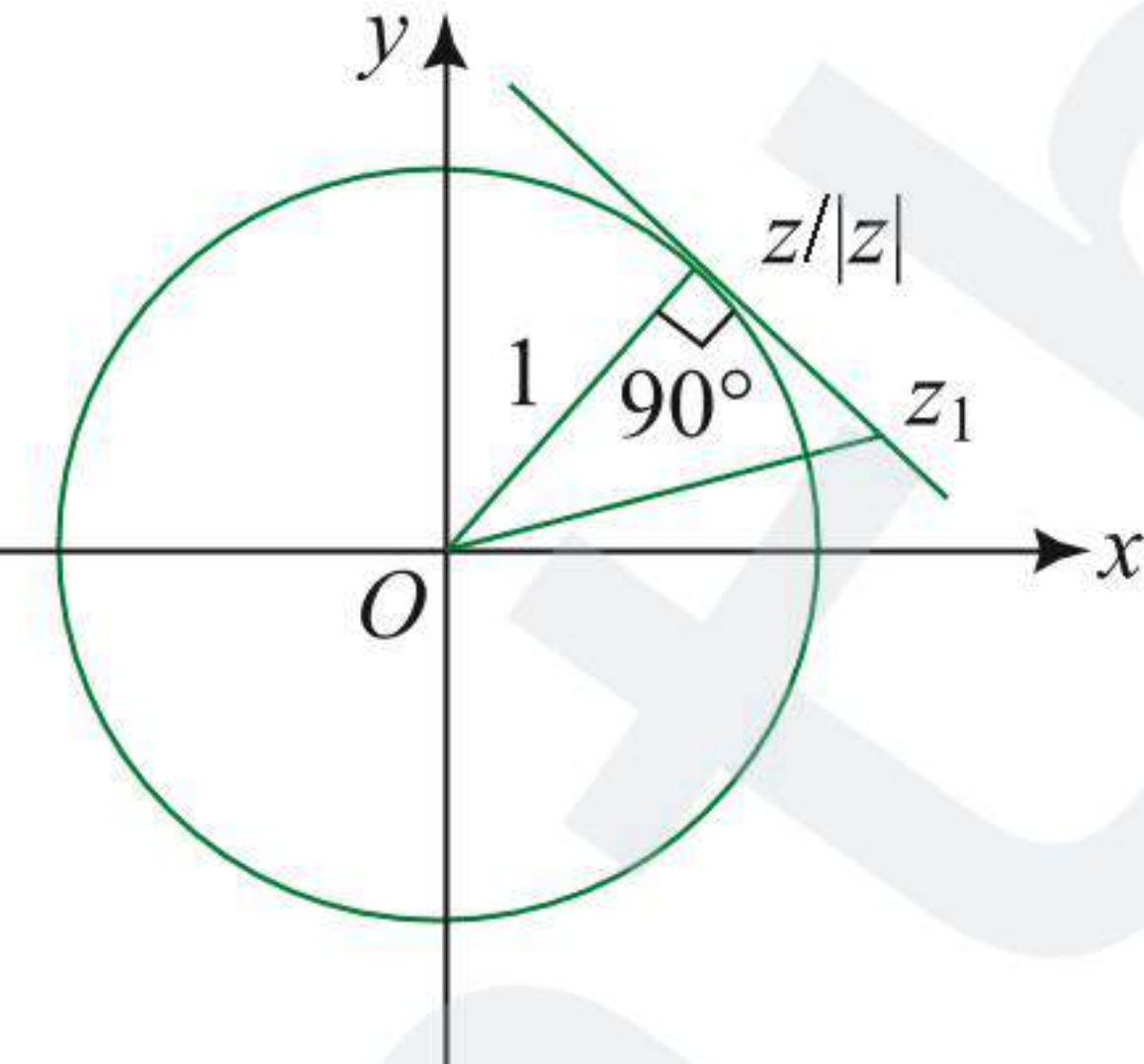


$$\begin{aligned} 2(\bar{z}_2 - \bar{z}_1) &= z_3 \left(\frac{\bar{z}_1}{z_1} - \frac{\bar{z}_2}{z_2} \right) \\ \Rightarrow \frac{2r^2(z_1 - z_2)}{z_1 z_2} &= z_3 r^2 \left(\frac{z_2^2 - z_1^2}{z_1^2 z_2^2} \right) \quad [\because |z_1|^2 = |z_2|^2 = r^2] \\ \Rightarrow z_3 &= \frac{2z_1 z_2}{z_2 + z_1} \end{aligned}$$

80. (2) Given that

$$\arg \left(\frac{z_1 - \frac{z}{|z|}}{\frac{z}{|z|}} \right) = \frac{\pi}{2}$$

and $\left| \frac{z}{|z|} - z_1 \right| = 3$

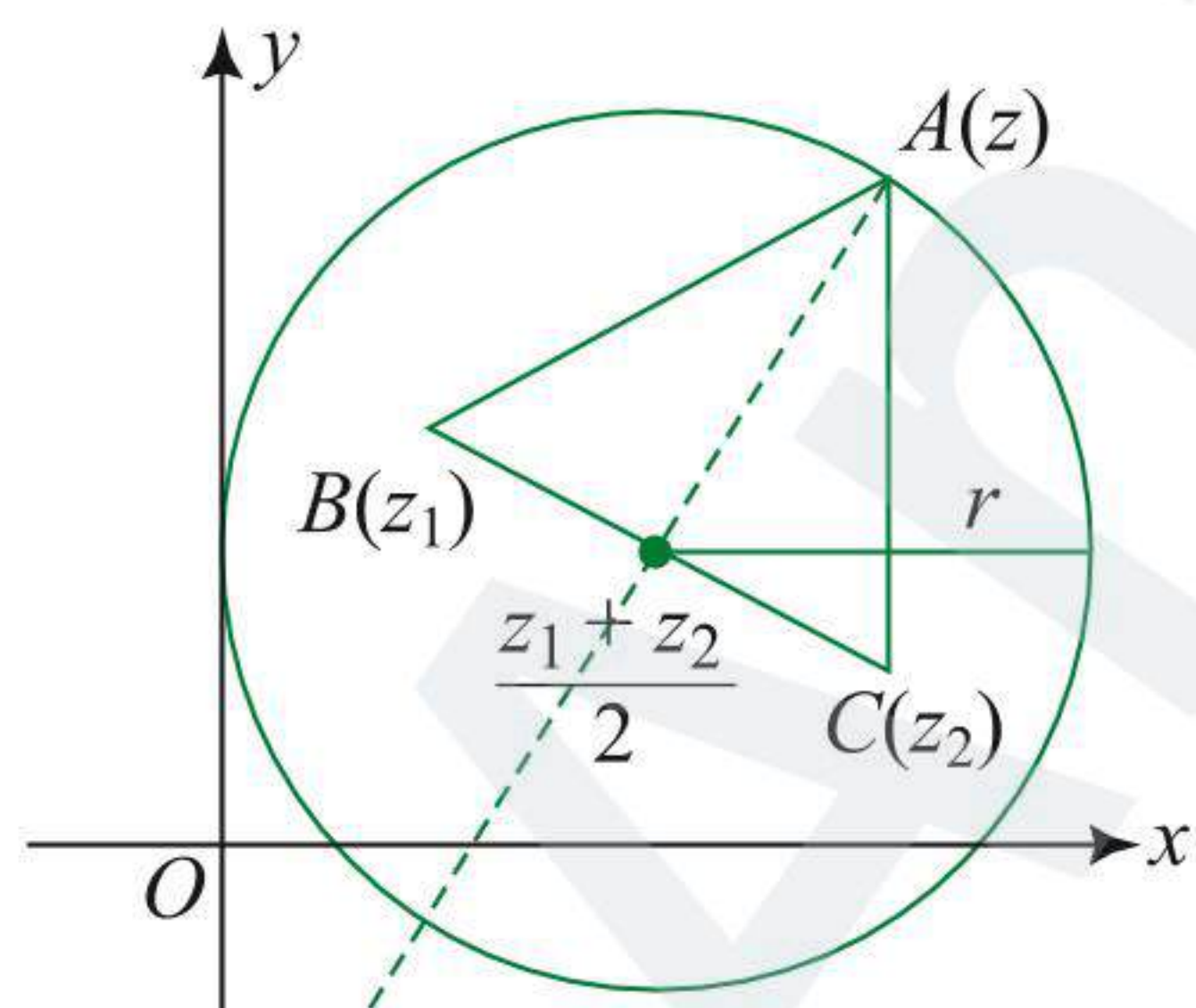


from which we can establish the above geometry.

From the diagram,

$$|z_1| = \sqrt{9+1} = \sqrt{10}$$

81. (2)



By the given conditions, the area of the triangle ABC is given by

$$(1/2)|z_1 - z_2|r.$$

82. (1) Given $|z| = 1$.

$$\text{and } |z^2 - az + 1| = 1$$

$$\Rightarrow \left| z \left| z - a + \frac{1}{z} \right| \right| = 1$$

$$\Rightarrow \left| a - \left(z + \frac{1}{z} \right) \right| = 1 \quad (1)$$

Now, $z = \cos \theta + i \sin \theta$

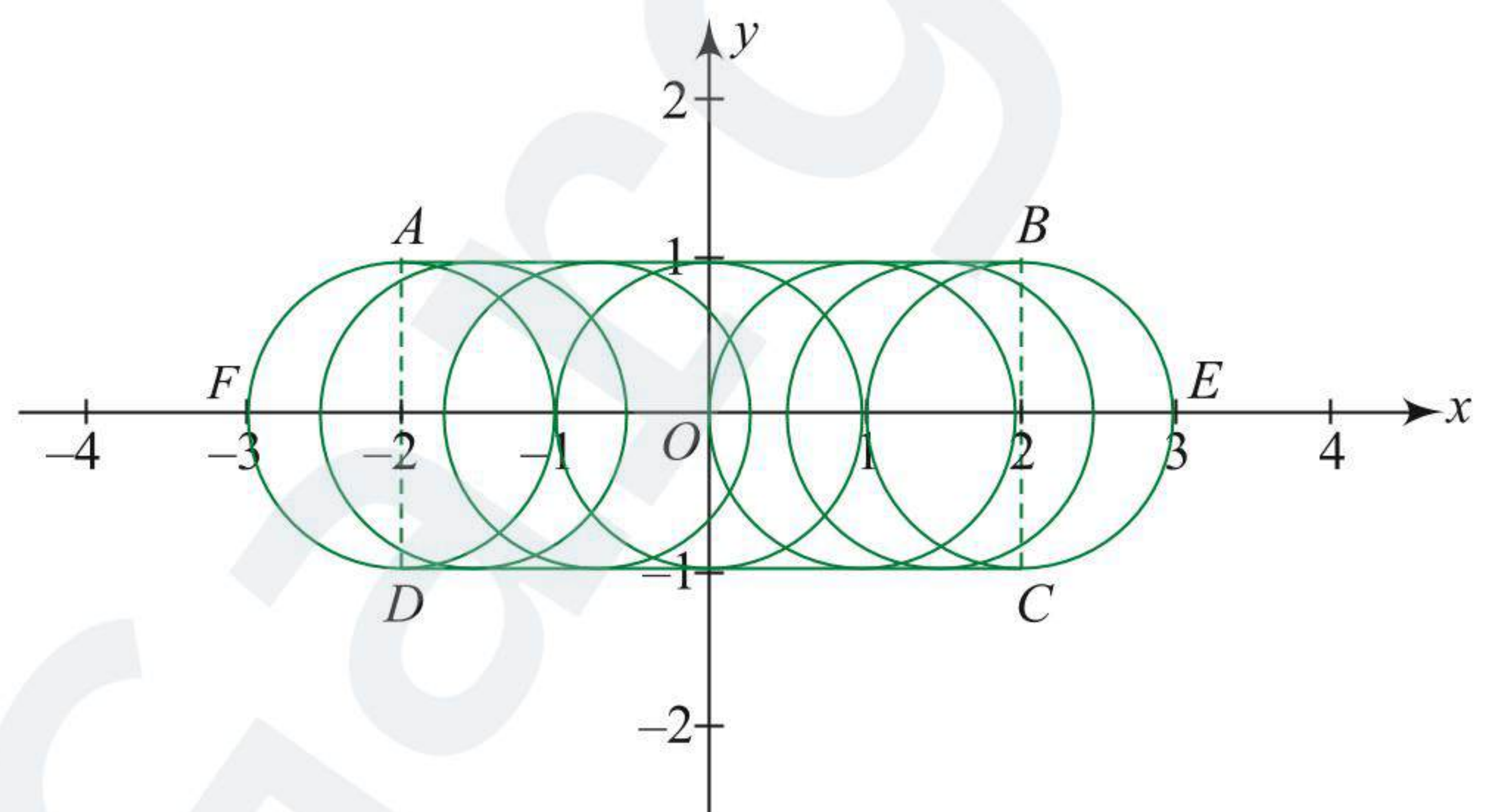
$$\Rightarrow z + \frac{1}{z} = 2 \cos \theta$$

So, (1) reduces to

$$|a - 2 \cos \theta| = 1$$

So, locus of a is circle whose center is $(2 \cos \theta, 0)$ and radius is 1.

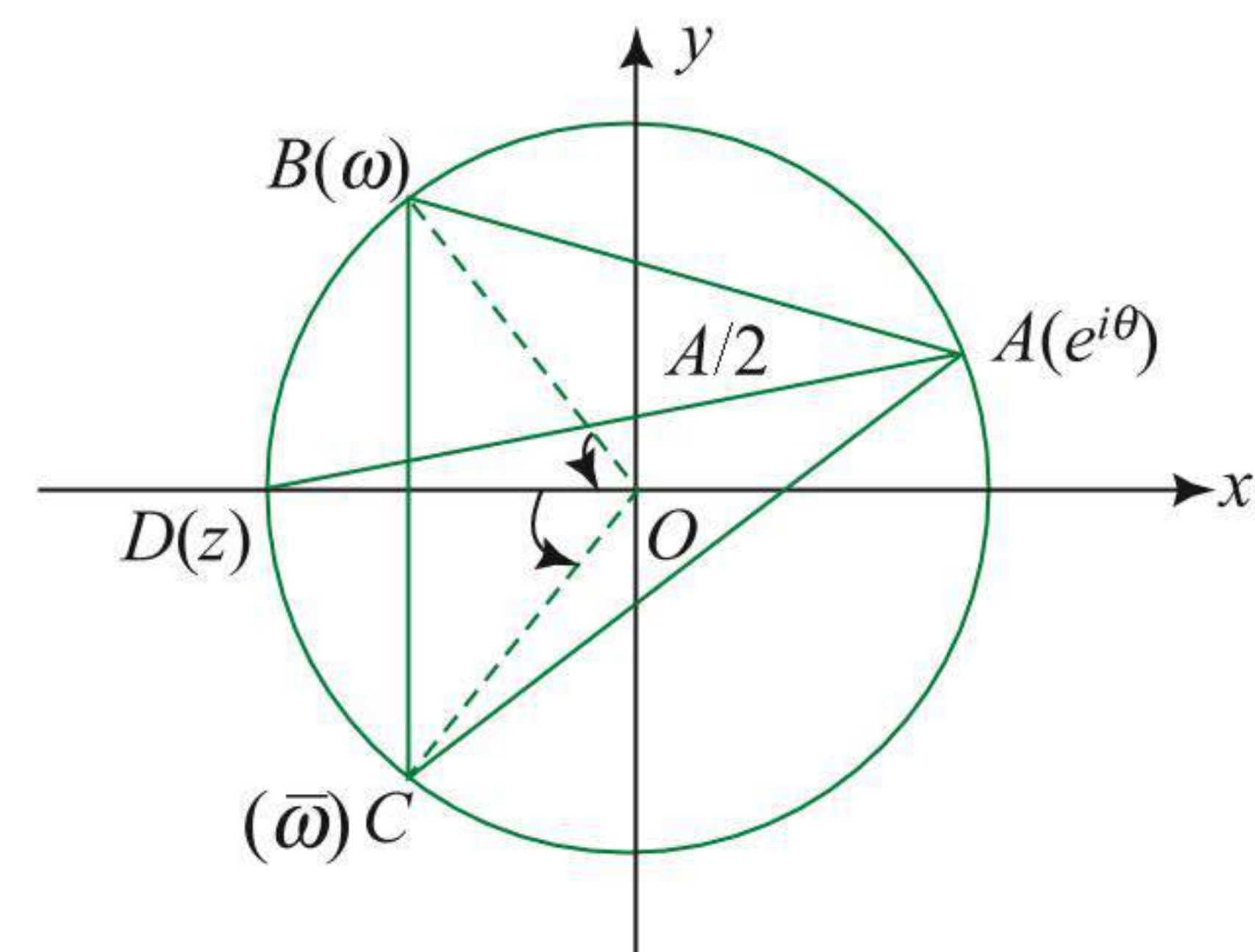
Since, $-2 \leq 2 \cos \theta \leq 2$, we have following region traced by the locus of a .



As shown in the figure the set S consists of the locus of units circles centered at a point on the line segment with end points at -2 and 2 on the Argand plane. Therefore, the area S consists of the area of two semi circles and area of rectangle $ABCD$.

So, the required area is $\pi + 8$.

83. (4) Clearly,



$$\angle DOB = \angle COD = A$$

$$\Rightarrow z = \omega e^{iA} \text{ and } \bar{\omega} = z e^{iA} \quad (\text{Applying rotation about } O)$$

$$\Rightarrow z^2 = \omega \bar{\omega} = 1$$

$$\Rightarrow z = -1$$

(As A and D are on opposite sides of BC)

84. (4) We have,

$$e^{\frac{2\pi r i}{p}} = e^{\frac{2\pi m i}{q}}$$

$$r = 0, 1, \dots, p-1$$

$$m = 0, 1, \dots, q-1$$

This is possible iff $r = m = 0$ but for $r = m = 0$ we get 1 which is not an imaginary number.

$$85. (1) \left(\frac{1+ia}{1-ia} \right)^4 = z$$

$$\text{or } \left| \frac{1+ia}{1-ia} \right|^4 = |z|$$

$$\text{or } \left| \frac{a-i}{a+i} \right|^4 = 1$$

$$\text{or } |a - i| = |a + i|$$

Therefore, a lies on the perpendicular bisector of i and $-i$, which is the real axis. Hence, all the roots are real. Obviously roots are distinct.

86. (1) We have

$$\log z + \log z^2 + \log z^3 + \dots + \log z^n = 0$$

$$\Rightarrow \log (z z^2 z^3 \dots z^n) = 0$$

$$\Rightarrow \log \left(z^{\frac{n(n+1)}{2}} \right) = 0$$

$$\text{or } z^{\frac{n(n+1)}{2}} = 1$$

$$\text{or } z = 1^{\frac{2}{n(n+1)}}$$

$$= (\cos 0^\circ + i \sin 0^\circ)^{\frac{2}{n(n+1)}}$$

$$= (\cos 2m\pi + i \sin 2m\pi)^{\frac{2}{n(n+1)}}, m = 0, 1, 2, 3, \dots$$

$$= \cos \frac{4m\pi}{n(n+1)} + i \sin \frac{4m\pi}{n(n+1)}, m = 0, 1, 2, \dots$$

87. (4) The equation $z^n = (z + 1)^n$ will have exactly $n - 1$ roots. We have,

$$\left(\frac{z+1}{z} \right)^n = 1 \quad \text{or} \quad \left| \frac{z+1}{z} \right| = 1$$

$$\text{or } |z + 1| = |z|$$

Therefore, z lies on the right bisector of the segment connecting the points $(0, 0)$ and $(-1, 0)$. Thus, $\text{Re}(z) = -1/2$. Hence, roots are collinear and will have their real parts equal to $-1/2$.

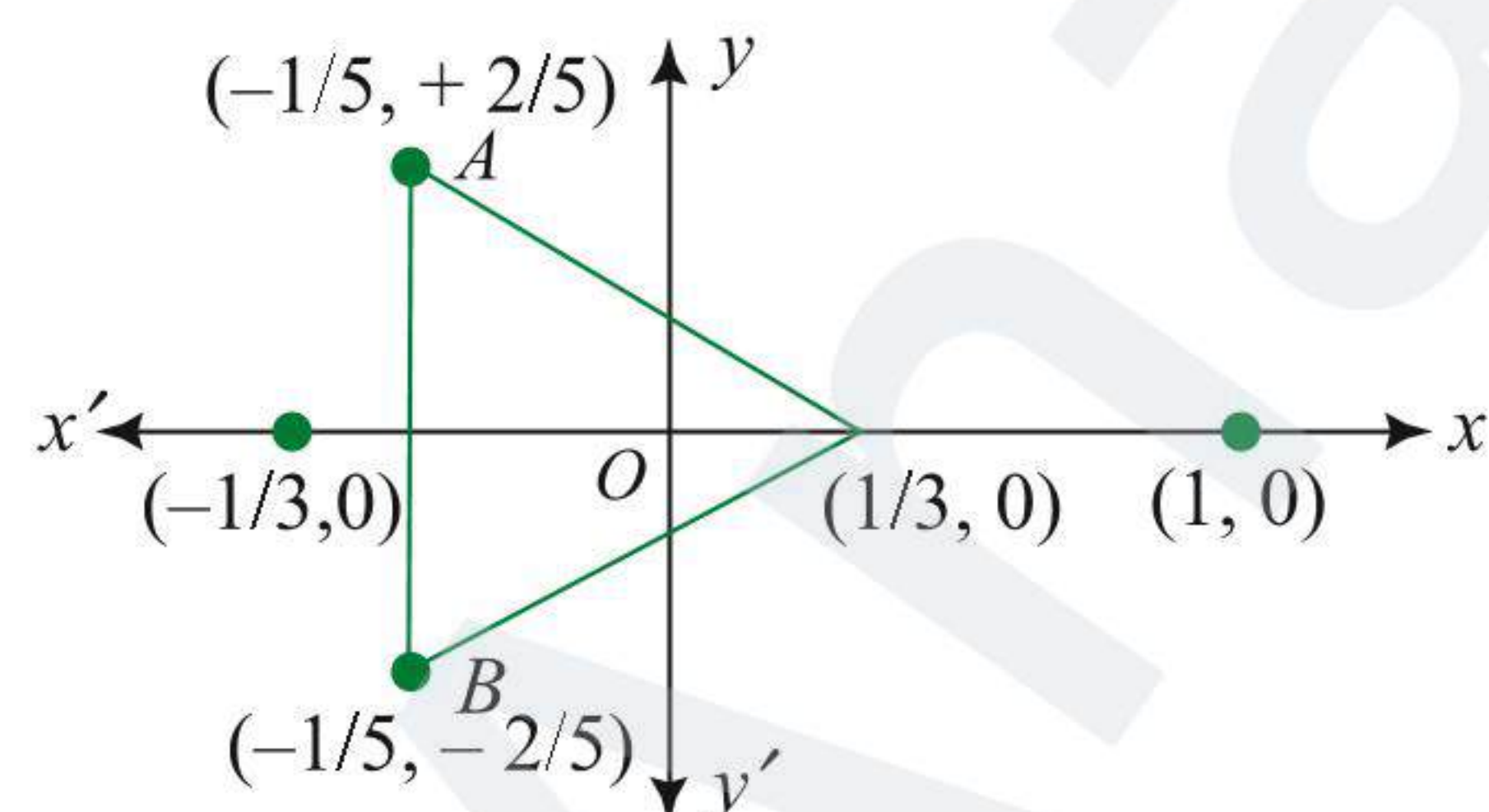
Hence, sum of the real parts of roots is $(-1/2)(n-1)$.

88. (3) $\left(\frac{z+1}{z} \right)^4 = 16$

$$\Rightarrow \frac{z+1}{z} = \pm 2, \pm 2i$$

The roots are $1, -1/3, (-1/5 - (2/5)i)$, and $(-1/5 + (2/5)i)$.

Note that $(-1/3, 0)$ and $(1, 0)$ are equidistant from $(1/3, 0)$ and since it lies on the perpendicular bisector of AB , it will be equidistant from A and B also.



89. (1) Given,

$$z_k = 1 + a + a^2 + \dots + a^{k-1} = \frac{1-a^k}{1-a}$$

$$\Rightarrow z_k - \frac{1}{1-a} = -\frac{a^k}{1-a}$$

$$\Rightarrow \left| z_k - \frac{1}{1-a} \right| = \frac{|a|^k}{|1-a|} < \frac{1}{|1-a|} \quad [\because |a| < 1]$$

Hence, z_k lies within the circle.

$$\therefore \left| z - \frac{1}{1-a} \right| = \frac{1}{|1-a|}$$

Multiple Correct Answers Type

1. (1), (3)

Clearly, we have to find it for real z . Let $z = x$. Then,

$$|x - w| = |x - w^2| = |w - w^2|$$

$$\Rightarrow \left(x + \frac{1}{2} \right)^2 + \frac{3}{4} = \left| \frac{-1 + \sqrt{3}i}{2} - \frac{-1 - \sqrt{3}i}{2} \right|^2 = 3 \Rightarrow x + \frac{1}{2} = \pm \frac{3}{2}$$

$$\Rightarrow x = 1, -2$$

2. (2), (3)

$$\text{amp}(z_1 z_2) = 0$$

$$\Rightarrow \text{amp } z_1 + \text{amp } z_2 = 0$$

$$\therefore \text{amp } z_1 = -\text{amp } z_2 = \text{amp } \bar{z}_2$$

Since $|z_1| = |z_2|$, we get $|z_1| = |\bar{z}_2|$.

So, $z_1 = \bar{z}_2$. Also, $z_1 z_2 = \bar{z}_2 z_2 = |z_2|^2 = 1$ because $|z_2| = 1$.

3. (1), (2), (3), (4)

$$\sqrt{5-12i} = \sqrt{(3-2i)^2} = \pm(3-2i)$$

$$\sqrt{-5-12i} = \sqrt{(2-3i)^2} = \pm(2-3i)$$

$$\Rightarrow z = \sqrt{5-12i} + \sqrt{-5-12i}$$

$$= -1-i, -5+5i, 5-5i, 1+i$$

Therefore, principal values of $\arg z$ are $-3\pi/4, 3\pi/4, -\pi/4, \pi/4$.

4. (1), (3)

$$(-i)^{1/3} = (i^3)^{1/3} = i, i\omega, i\omega^2$$

$$\text{where } \omega = \frac{-1 + \sqrt{3}i}{2}$$

Hence, values are $i, (-\sqrt{3}-i)/2, (\sqrt{3}-i)/2$.

5. (1), (3), (4)

$$\text{We have } a^3 + b^3 - 8c^3 + 6abc = 0$$

$$\Rightarrow a^3 + b^3 + (-2c)^3 - 3ab(-2c) = 0$$

$$\Rightarrow (a+b-2c)(a+b\omega-2c\omega^2)(a+b\omega^2-2c\omega) = 0$$

6. (3), (4)

$$|z_1 + z_2| = |z_1| = |z_2|$$

Thus, angle between z_1 and z_2 is 120° .

$$\therefore \frac{z_1}{z_2} = \omega \quad \text{or} \quad \omega^2$$

7. (1), (3)

$$p + q + r = a + b\omega + c\omega^2 + b + c\omega + a\omega^2 + c + a\omega + b\omega^2$$

$$\therefore p + q + r = (a + b + c)(1 + \omega + \omega^2) = 0 \quad (1)$$

p, q, r lie on the circle $|z| = 2$, whose circumcenter is origin. Also, $(p + q + r)/3 = 0$. Hence the centroid coincides with circumcenter. So, the triangle is equilateral.

Now, $(p + q + r)^2 = 0$.

$$\Rightarrow p^2 + q^2 + r^2 = -2pqr \left[\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right]$$

$$= -2pqr \left[\frac{1}{a+b\omega+c\omega^2} + \frac{1}{b+c\omega+a\omega^2} + \frac{1}{c+a\omega+b\omega^2} \right]$$

$$= -2pqr \left[\frac{1}{\omega^2(a\omega+b\omega^2+c)} + \frac{1}{\omega(b\omega^2+c+a\omega)} + \frac{1}{c+a\omega+b\omega^2} \right]$$

$$= \frac{-2pqr}{a\omega + b\omega^2 + c} \left[\frac{1}{\omega^2} + \frac{1}{\omega} + \frac{1}{1} \right] = 0 \quad (2)$$

Hence, $p^2 + q^2 + r^2 = 2(pq + qr + rp)$

8. (3), (4)

We have,

$$x^2 + x + 1 = (x - \omega)(x - \omega^2)$$

Since $f(x)$ is divisible by $x^2 + x + 1$, $f(\omega) = 0$, $f(\omega^2) = 0$, so

$$P(\omega^3) + \omega Q(\omega^3) = 0 \Rightarrow P(1) + \omega Q(1) = 0 \quad (1)$$

$$P(\omega^6) + \omega^2 Q(\omega^6) = 0 \Rightarrow P(1) + \omega^2 Q(1) = 0 \quad (2)$$

Solving Eqs. (1) and (2), we obtain

$$P(1) = 0 \text{ and } Q(1) = 0$$

Therefore, both $P(x)$ and $Q(x)$ are divisible by $x - 1$. Hence, $P(x^3)$ and $Q(x^3)$ are divisible by $x^3 - 1$ and so by $x - 1$. Since $f(x) = P(x^3) + xQ(x^3)$, we get $f(x)$ is divisible by $x - 1$.

9. (1), (3), (4)

Let $z = c$ be a real root. Then,

$$\alpha c^2 + c + \bar{\alpha} = 0 \quad (1)$$

Putting $\alpha = p + iq$, we have

$$(p + iq)c^2 + c + p - iq = 0$$

$$\Rightarrow pc^2 + c + p = 0 \text{ and } qc^2 - q = 0$$

$$\Rightarrow c = \pm 1 \quad (\because q \neq 0)$$

$$\therefore (1) \Rightarrow \alpha \pm 1 + \bar{\alpha} = 0$$

Also, $|c| = 1$

10. (1), (2), (4)

Let $z = \alpha$ be a real root. Then,

$$\alpha^3 + (3 + 2i)\alpha + (-1 + ia) = 0$$

$$\Rightarrow (\alpha^3 + 3\alpha - 1) + i(a + 2\alpha) = 0$$

$$\Rightarrow \alpha^3 + 3\alpha - 1 = 0 \text{ and } \alpha = -a/2$$

$$\Rightarrow -\frac{a^3}{8} - \frac{3a}{2} - 1 = 0$$

$$\Rightarrow a^3 + 12a + 8 = 0$$

$$\text{Let } f(a) = a^3 + 12a + 8.$$

$$f'(a) = 3a^2 + 12 > 0.$$

So, $f(a)$ cuts x -axis only once.

Now, $f(-1) < 0$, $f(0) > 0$,

$$f(-2) < 0, f(1) > 0 \text{ and } f(3) > 0$$

Hence, $a \in (-1, 0)$ or $a \in (-2, 1)$ or $a \in (-2, 3)$.

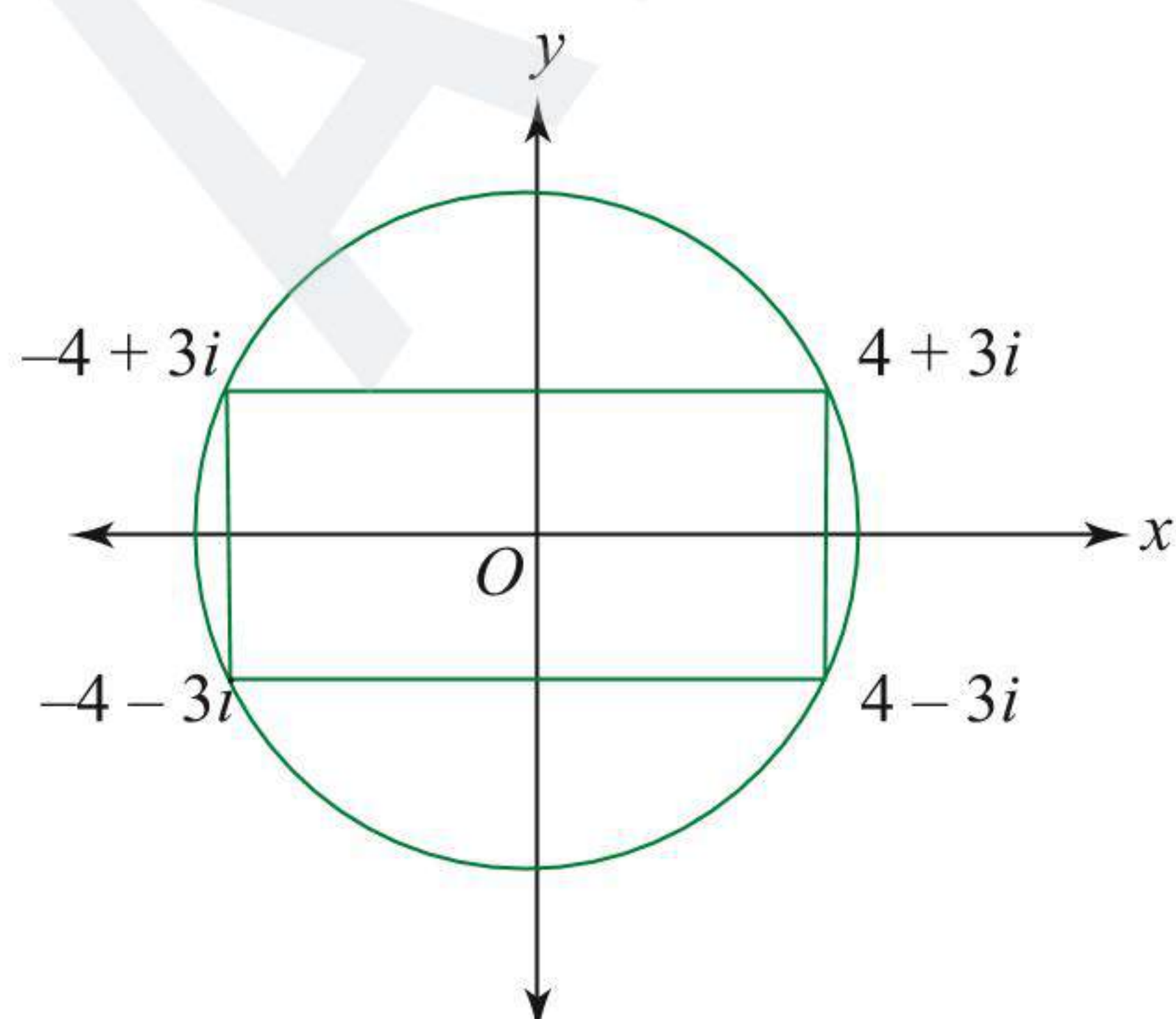
11. (1), (2), (3)

Let $z = x + iy$ where x, y satisfy the given equation. Hence,

$$(x^2 + y^2)(x^2 - y^2) = 175$$

$$\Rightarrow x^2 + y^2 = 25 \text{ and } x^2 - y^2 = 7 \text{ (as all other possibilities will give non-integral solutions)}$$

Hence, possible values of z will be $4 + 3i$, $4 - 3i$, $-4 + 3i$, and $-4 - 3i$. Clearly, it will form a rectangle having length of the diagonal 10.



From the diagram, options (1), (2), (3) are correct.

12. (1), (3)

Triangle ABC is equilateral. Hence,

$$z^2 + (-z)^2 + (1 - z)^2 = z(-z) + z(1 - z) + (-z)(1 - z)$$

$$\text{or } 3z^2 - 2z + 1 = -z^2$$

$$\text{or } 4z^2 - 2z + 1 = 0$$

$$\text{Sum of roots} = \frac{1}{2}$$

$$\text{and Product of roots} = \frac{1}{4}$$

13. (1), (4)

If $|z - 3| = |z - 1|$, then z lies on perpendicular bisector of line segment joining '1' and '3'. So, $\text{Re}(z) = 2$.

If $|z - 3| = |z - 5|$, then z lies on perpendicular bisector of line segment joining '3' and '5'. So, $\text{Re}(z) = 4$.

14. (1), (4)

$$|z_1^2 - z_2^2| = |\bar{z}_1^2 + \bar{z}_1^2 - 2\bar{z}_1\bar{z}_2|$$

$$\text{or } |z_1 - z_2||z_1 + z_2| = |\bar{z}_1 - \bar{z}_2|^2$$

$$\text{or } |z_1 + z_2| = |\bar{z}_1 - \bar{z}_2|$$

$$|z_1 + z_2| = |z_1 - z_2|$$

$$\Rightarrow \left| \frac{z_1}{z_2} + 1 \right| = \left| \frac{z_1}{z_2} - 1 \right|$$

$$\Rightarrow \frac{z_1}{z_2} \text{ lies on } \perp \text{ bisector of } 1 \text{ and } -1$$

$$\Rightarrow \frac{z_1}{z_2} \text{ lies on imaginary axis}$$

$$\Rightarrow \frac{z_1}{z_2} \text{ is purely imaginary}$$

$$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \pm \frac{\pi}{2}$$

$$|\arg(z_1) - \arg(z_2)| = \frac{\pi}{2}$$

15. (1), (2), (3)

$$|z_1| = |z_2| = 1$$

$$\Rightarrow a^2 + b^2 = c^2 + d^2 = 1 \quad (1)$$

$$\text{and } \text{Re}(z_1 \bar{z}_2) = 0$$

$$\Rightarrow \text{Re}\{(a + ib)(c + id)\} = 0$$

$$\Rightarrow ac + bd = 0 \quad (2)$$

Now, from Eqs. (1) and (2), we get

$$a^2 + b^2 = 1$$

$$\Rightarrow a^2 + \frac{a^2 c^2}{d^2} = 1$$

$$\Rightarrow a^2 = d^2 \quad (3)$$

$$\text{Also, } c^2 + d^2 = 1$$

$$\Rightarrow c^2 + \frac{a^2 c^2}{b^2} = 1$$

$$\Rightarrow b^2 = c^2 \quad (4)$$

$$|\omega_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1 \quad [\text{From (1) and (4)}]$$

$$\text{and } |\omega_2| = \sqrt{b^2 + d^2} = \sqrt{c^2 + d^2} = 1 \quad [\text{from (1) and (4)}]$$

$$\text{Further } \text{Re}(\omega_1 \bar{\omega}_2) = \text{Re}\{(a + ic)(b - id)\}$$

$$= ab + cd$$

$$= ab - \frac{ac^2}{b}$$

[From (2)]

$$= \frac{ab^2 - ac^2}{b} = 0 \quad [\text{from (4)}]$$

$$\text{Also, } \text{Im}(\omega_1 \bar{\omega}_2) = bc - ad$$

$$= bc - a \left(-\frac{ac}{b} \right) = \frac{(a^2 + b^2)c}{b} = \frac{c}{b} = \pm 1 \neq 0$$

$$\therefore |\omega_1| = 1, |\omega_2| = 1 \text{ and } \text{Re}(\omega_1 \bar{\omega}_2) = 0$$

16. (1), (4)

Let $z_1 = a + ib$, $a > 0$ and $b \in \mathbb{R} : z_2 = c + id$, $d < 0$, $c \in \mathbb{R}$

Given $|z_1| = |z_2|$

$$\Rightarrow a^2 + b^2 = c^2 + d^2$$

$$\Rightarrow a^2 - c^2 = d^2 - b^2 \quad \dots(1)$$

$$\text{Now, } \frac{z_1 + z_2}{z_1 - z_2} = \frac{(a+c) + i(b+d)}{(a-c) + i(b-d)}$$

$$= \frac{[(a^2 - c^2) + (b^2 - d^2)] + i[(a-c)(b+d) - (a+c)(b-d)]}{(a-c)^2 + (b-d)^2}$$

This values will be purely imaginary as $a^2 - c^2 = d^2 - b^2$ or the value will be zero in case $a + c = b + d = 0$.

17. (1), (3)

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1 \bar{z}_2$$

$$\Rightarrow 5 - 2\sqrt{3} = 2 + 3 + 2|z_1||z_2|\cos(\theta_1 - \theta_2),$$

where $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

$$\Rightarrow -2\sqrt{3} = 2\sqrt{2}\sqrt{3}\cos(\theta_1 - \theta_2)$$

$$\Rightarrow \cos(\theta_1 - \theta_2) = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta_1 - \theta_2 = \frac{3\pi}{4}, \frac{5\pi}{4}$$

$$\Rightarrow \arg(z_1) - \arg(z_2) = \arg\left(\frac{z_1}{z_2}\right) = \frac{3\pi}{4}, \frac{5\pi}{4}$$

18. (2), (3)

$$z_3 = \frac{z_2(z_1 - z_4)}{\bar{z}_1 z_4 - 1}$$

$$\Rightarrow |z_3| = \frac{|z_2||z_1 - z_4|}{|\bar{z}_1 z_4 - 1|} = \frac{|z_1 - z_4|}{|\bar{z}_1 z_4 - 1|} \leq 1$$

$$\text{If and only if } |z_1 - z_4|^2 \leq |\bar{z}_1 z_4 - 1|^2$$

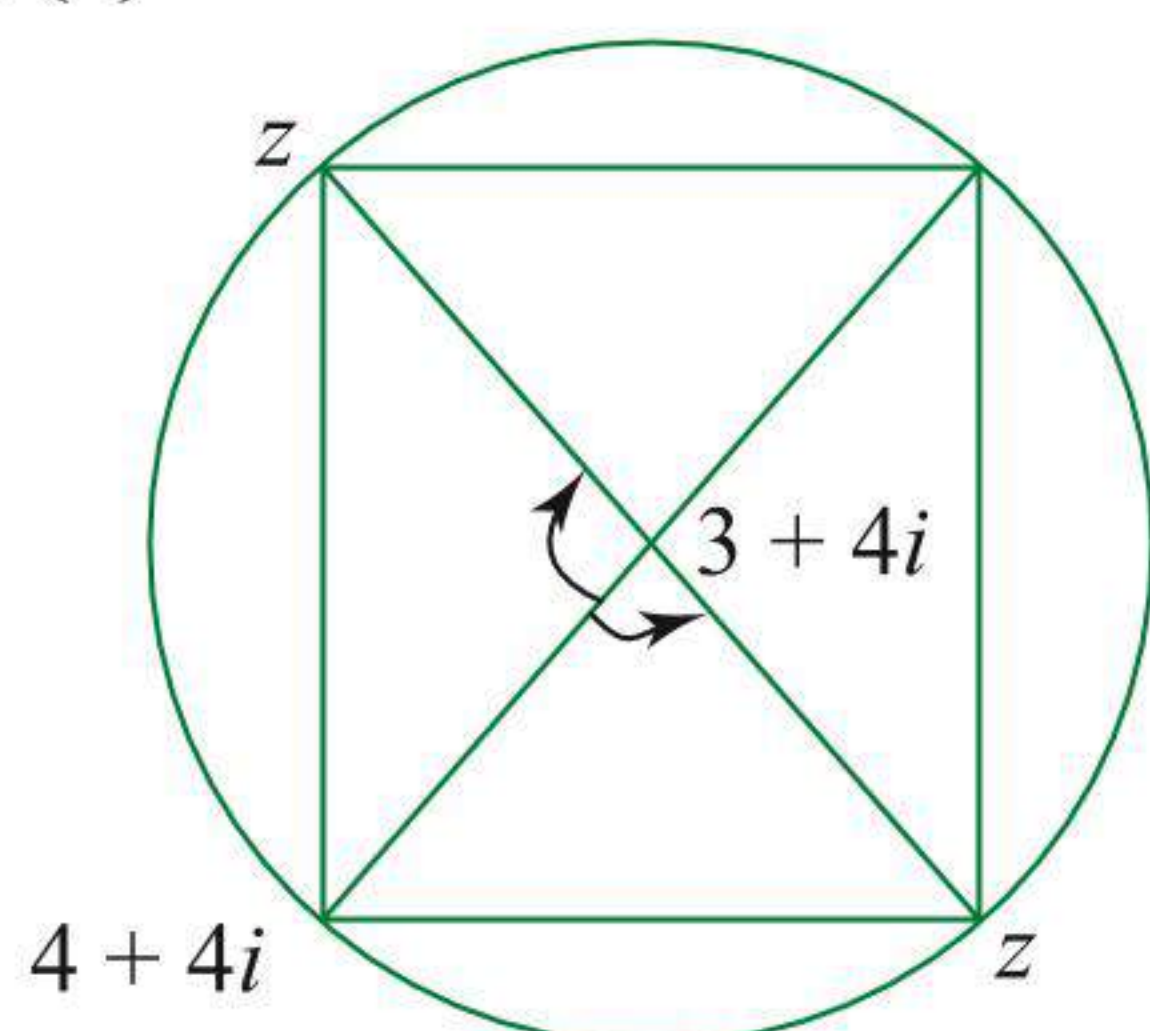
$$\Rightarrow |z_1|^2 + |z_4|^2 - \bar{z}_1 z_4 - z_1 \bar{z}_4 \leq |\bar{z}_1 z_4|^2 + 1 - \bar{z}_1 z_4 - z_1 \bar{z}_4$$

$$\Rightarrow |z_1|^2 + |z_4|^2 - |z_1|^2 |z_4|^2 - 1 \leq 0$$

$$\Rightarrow (|z_1|^2 - 1)(|z_4|^2 - 1) \geq 0$$

Since $|z_1| \leq 1$, we must have $|z_4| \leq 1$

19. (2), (3)



Clearly, the inscribed rectangle is a square. Let the adjacent vertex be z . Then,

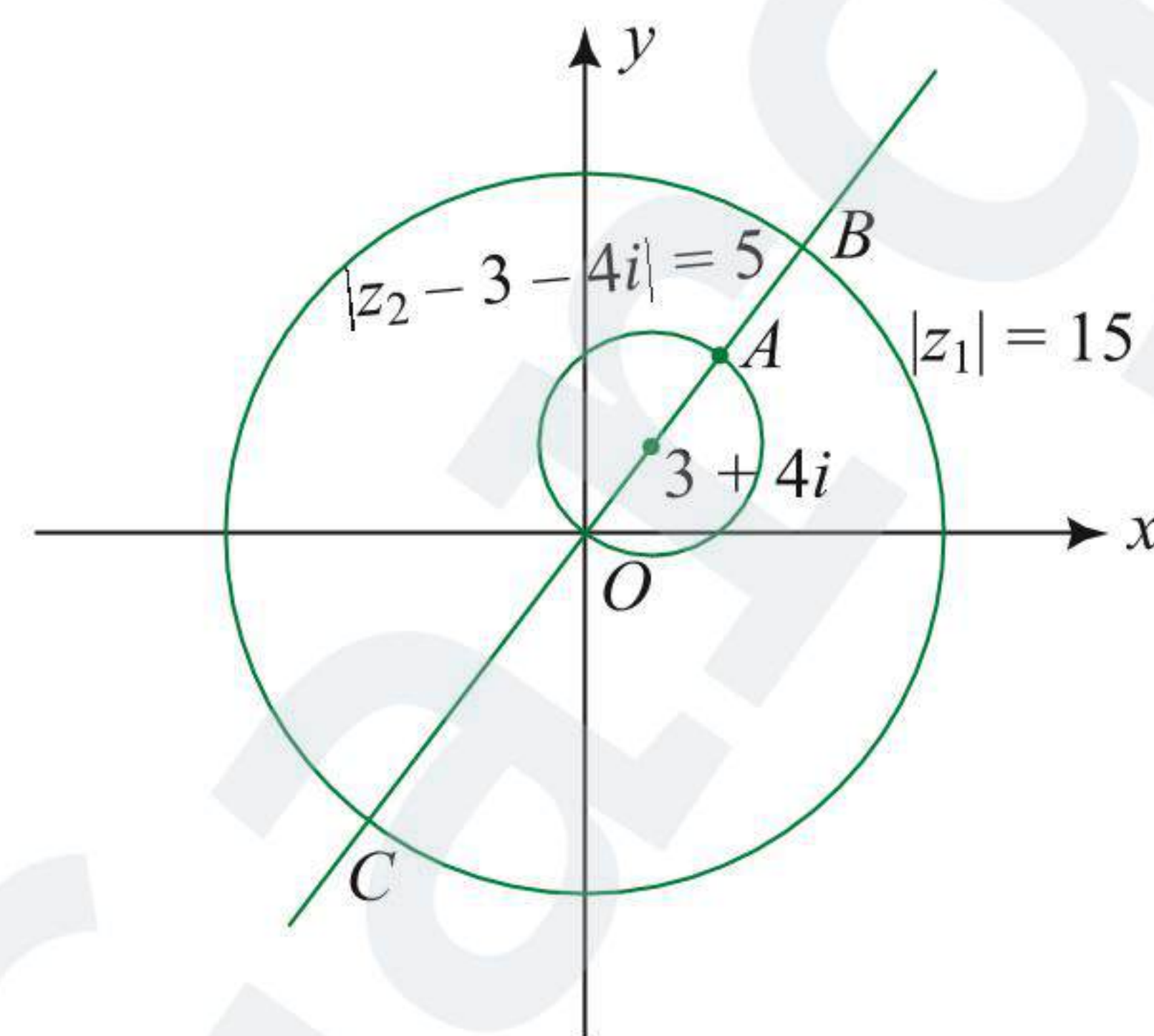
$$\frac{3+4i - (z)}{(3+4i) - (4+4i)} = e^{\pm i\pi/2} \quad (\text{by rotation about center})$$

$$\Rightarrow 3+4i - z = \pm i(-1)$$

$$\Rightarrow z = 3+4i \pm i = 3+5i \text{ or } 3+3i$$

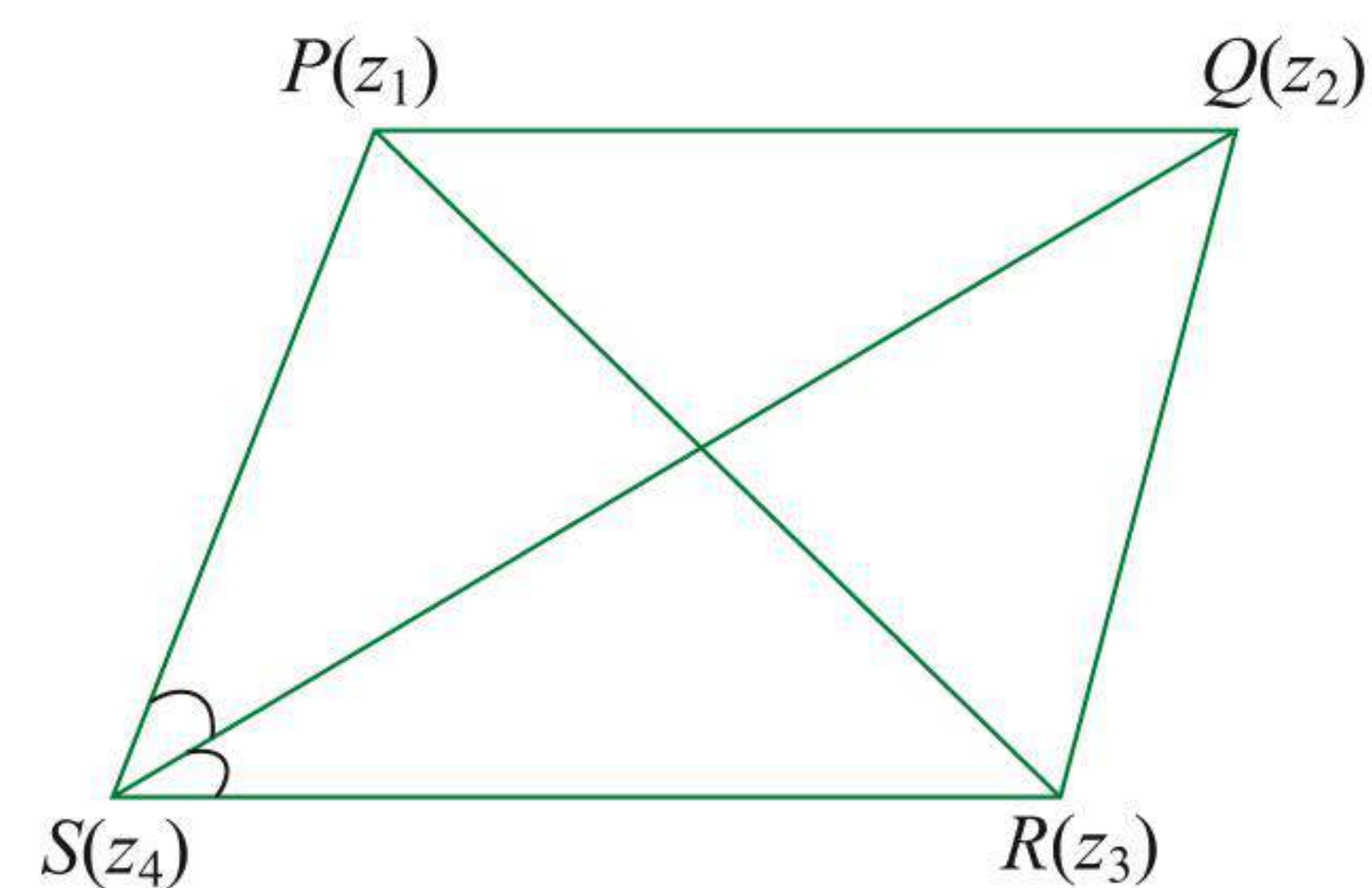
20. (1), (4)

We have, $|z_1| = 15$, $|z_2 - 3 - 4i| = 5$



Minimum value of $|z_1 - z_2|$ is $AB = OB - OA = 15 - 10 = 5$.
Maximum value of $|z_1 - z_2|$ is $CA = OA + OC = 10 + 15 = 25$.

21. (1), (2), (3), (4)



$$(1) PS \parallel QR \Rightarrow \arg\left(\frac{z_1 - z_4}{z_2 - z_3}\right) = 0 \Rightarrow \frac{z_1 - z_4}{z_2 - z_3} \text{ is purely real}$$

(2) Since diagonal bisect the angle

$$\Rightarrow \arg\left(\frac{z_1 - z_4}{z_2 - z_4}\right) = \arg\left(\frac{z_2 - z_4}{z_3 - z_4}\right)$$

(3) Diagonals of rhombus are perpendicular. Hence, $(z_1 - z_3)/(z_2 - z_4)$ is purely imaginary.

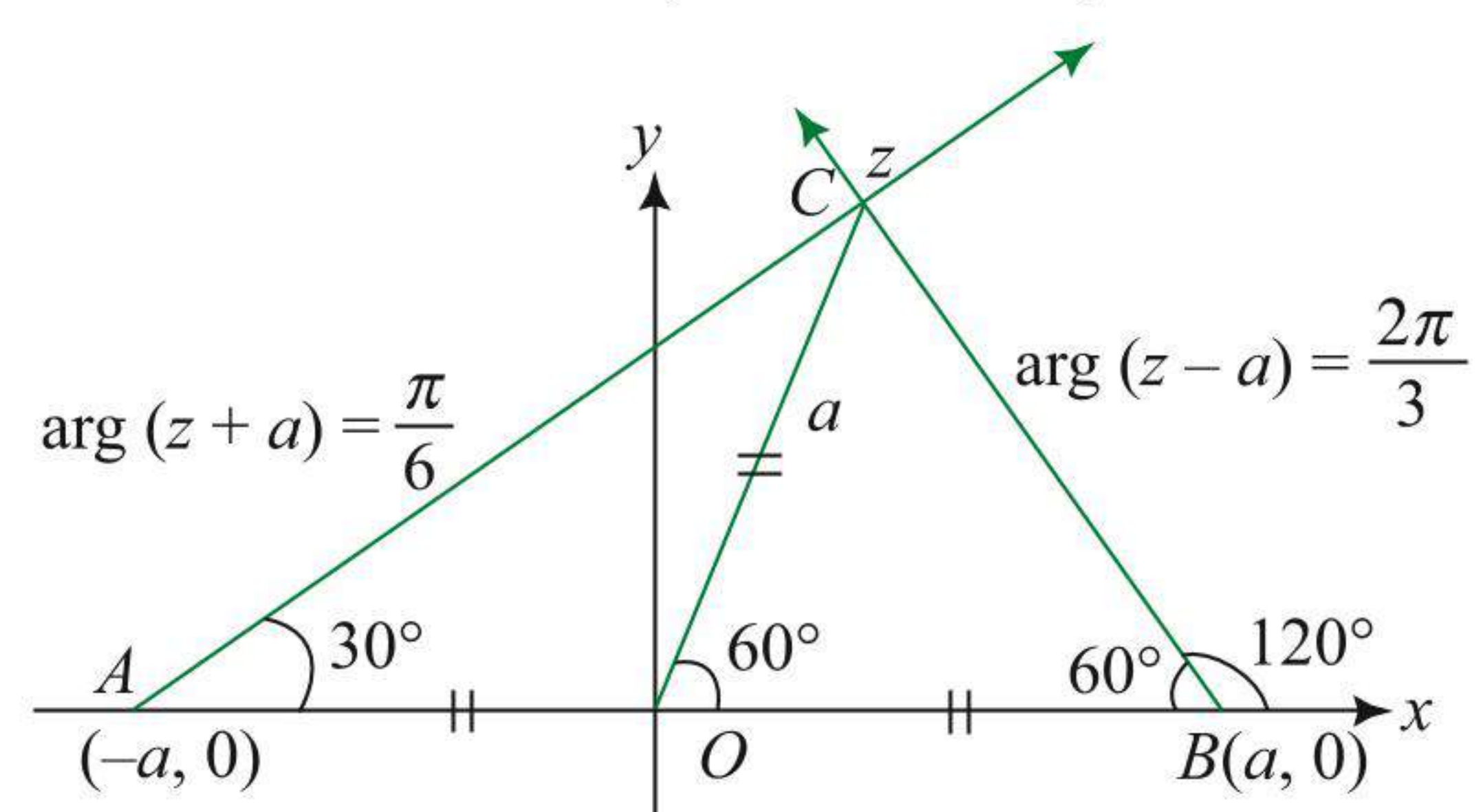
(4) Diagonals of rhombus are not necessarily equal.

Hence, $|z_1 - z_3| \neq |z_2 - z_4|$ not necessarily.

22. (1), (4)

Refer to figure, z lies on the point of intersection of the rays from A and B . $\angle ACB$ is a right angle and OBC is an equilateral triangle. Hence,

$$OC = a \Rightarrow z = a \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$$



23. (1), (2), (4)

We have

$$|z| = 1$$

This means that z lies on a circle whose centre is at origin and radius is 1.

$$\text{Also, } \arg(z-1) = \frac{2\pi}{3}$$

This implies that z lies on a ray emanating from $(1, 0)$ making an angle of $\frac{2\pi}{3}$ with positive real axis.

$$\Rightarrow \angle PQX = \frac{2\pi}{3}$$

$$\Rightarrow \angle PQO = \frac{\pi}{3}$$

and $OP = OQ$

Thus, triangle OPQ is equilateral.

$$\Rightarrow OP = OQ = PQ$$

$$\Rightarrow |z| = |z-1| = 1$$

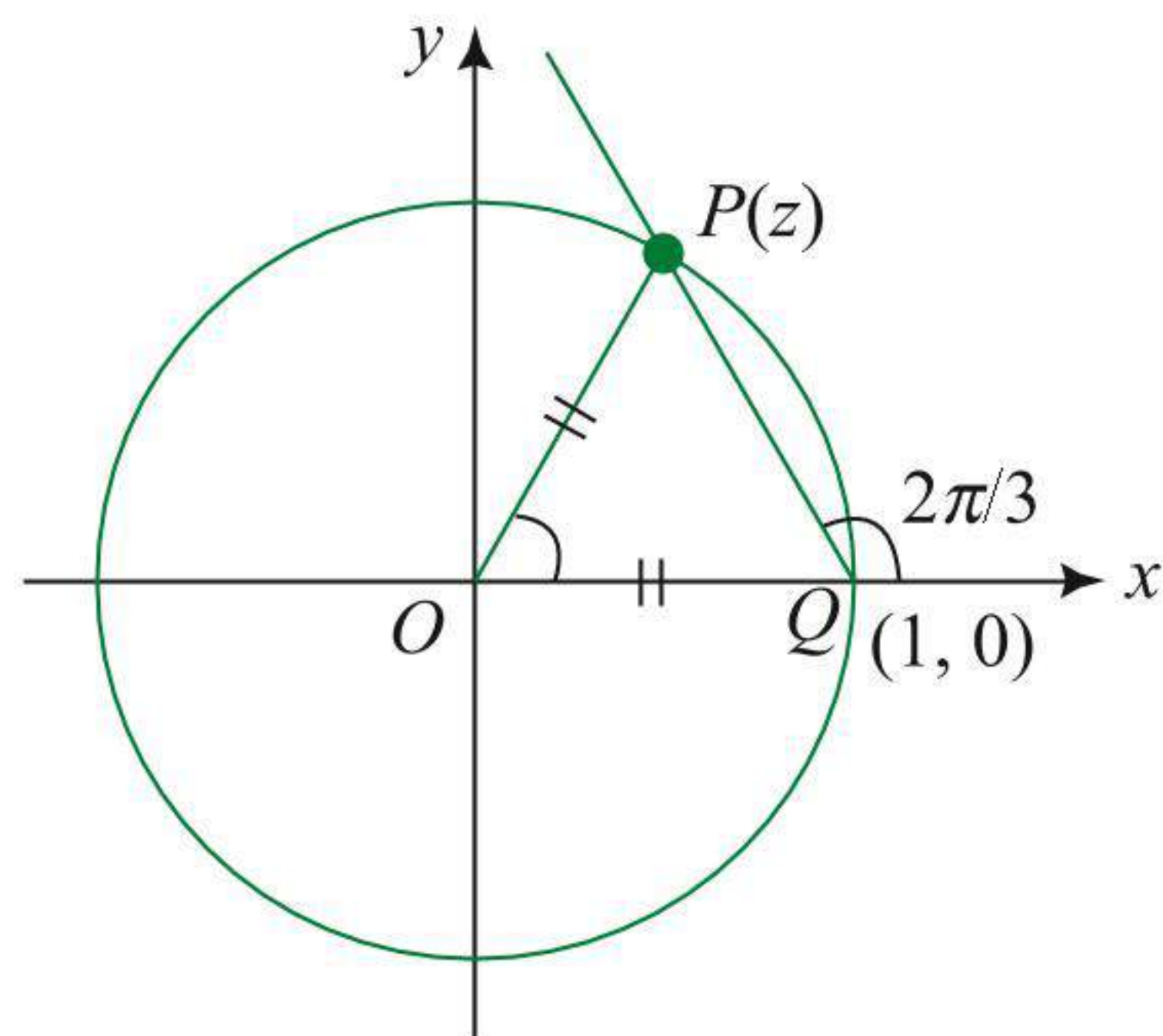
Also, $\arg(z) = 60^\circ$

$$\therefore z = \cos 60^\circ + i \sin 60^\circ = \frac{1+i\sqrt{3}}{2} = -\omega^2$$

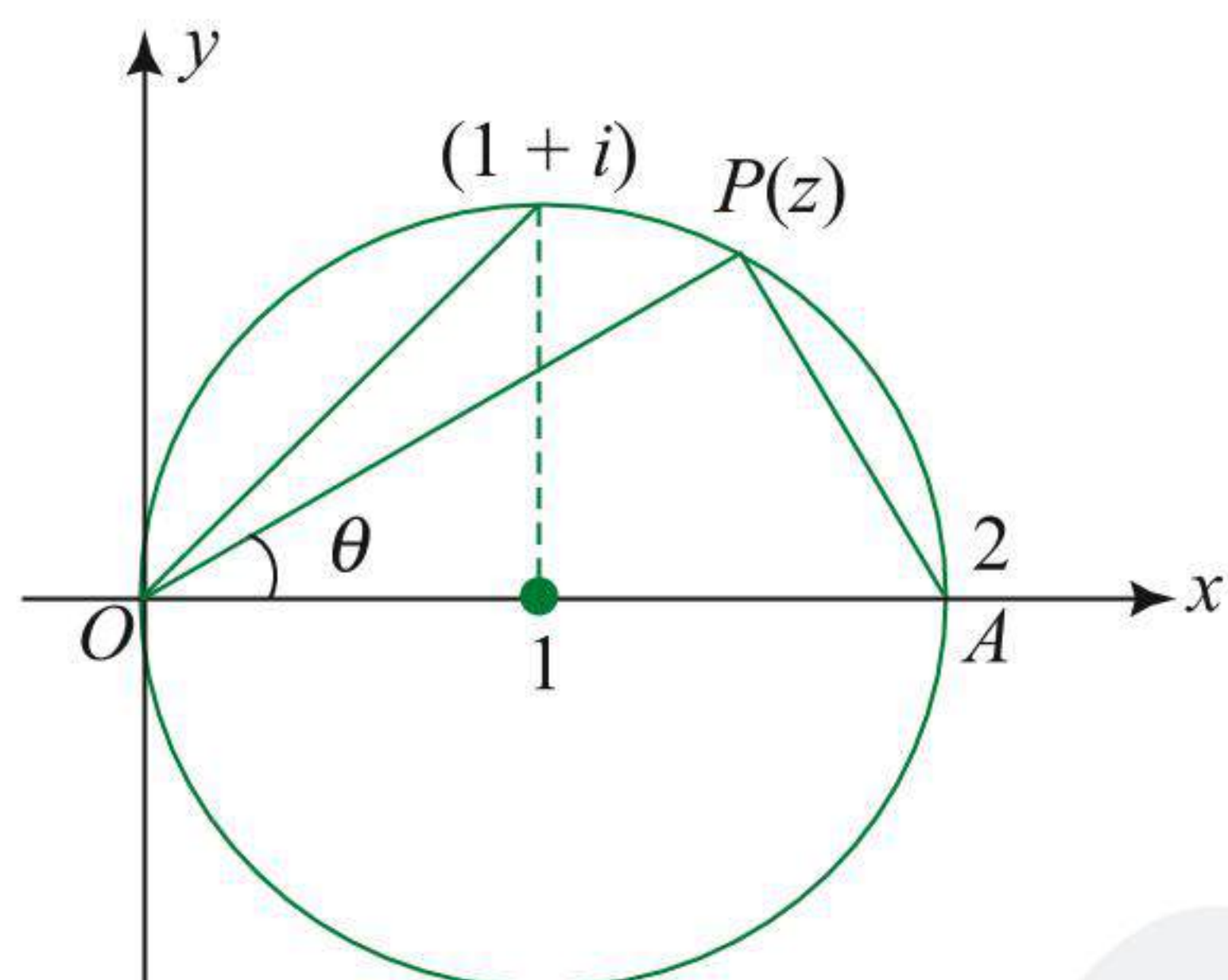
$$z^2 = \omega^4 = \omega$$

$$\Rightarrow z^2 + z = \omega - \omega^2 = \left(\frac{-1+i\sqrt{3}}{2}\right) - \left(\frac{-1-i\sqrt{3}}{2}\right) = i\sqrt{3}$$

Therefore, $z^2 + z$ is purely imaginary number.



24. (1), (2), (4)



Since $\arg((z-1-i)/z)$ is the angle subtended by the chord joining the points O and $1+i$ at the circumference of the circle $|z-1|=1$, so it is equal to $-\pi/4$. The line joining the points $z=0$ and $z=2+0i$ is the diameter.

$$\arg \frac{z-2}{z} = \pm \frac{\pi}{2}$$

$$\Rightarrow \frac{z-2}{z-0} \text{ is purely imaginary}$$

We have

$$\angle OPA = \frac{\pi}{2}$$

$$\Rightarrow \arg\left(\frac{2-z}{0-z}\right) = \frac{\pi}{2} \Rightarrow \frac{z-2}{z} = \frac{AP}{OP}i$$

Now in $\triangle OAP$,

$$\tan \theta = \frac{AP}{OP}$$

$$\text{Thus, } \frac{z-2}{z} = i \tan \theta$$

25. (1), (2), (4)

$$z_1 = 5 + 12i, |z_2| = 4$$

$$|z_1 + iz_2| \leq |z_1| + |z_2| = 13 + 4 = 17$$

$$\therefore |z_1 + (1+i)z_2| \geq ||z_1| - |1+i||z_2|| = 13 - 4\sqrt{2}$$

$$\therefore \min(|z_1 + (1+i)z_2|) = 13 - 4\sqrt{2}$$

$$\left|z_2 + \frac{4}{z_2}\right| \leq |z_2| + \frac{4}{|z_2|} = 4 + 1 = 5$$

$$\left|z_2 + \frac{4}{z_2}\right| \geq |z_2| - \frac{4}{|z_2|} = 4 - 1 = 3$$

$$\therefore \max \left| \frac{z_1}{z_2 + \frac{4}{z_2}} \right| = \frac{13}{3} \text{ and } \min \left| \frac{z_1}{z_2 + \frac{4}{z_2}} \right| = \frac{13}{5}$$

26. (1), (2), (4)

$$\text{Given, } \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow a^3 + b^3 + c^3 - 3abc = 0$$

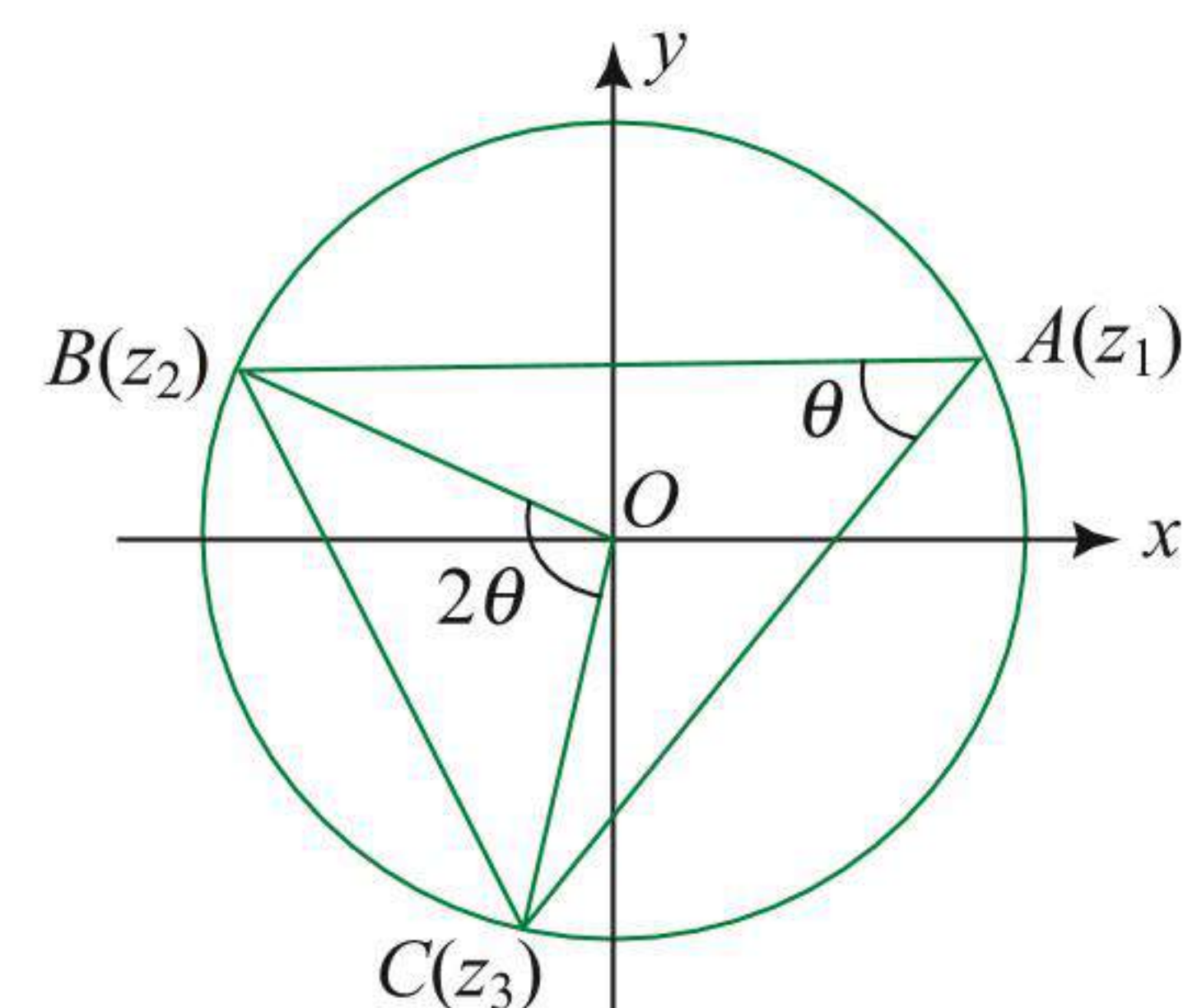
$$\Rightarrow (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$$

$$\Rightarrow \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$$

$$\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$$

$$\Rightarrow a = b = c \quad [\because a+b+c \neq 0, \because z_1 \neq 0, \therefore |z_1| = a \neq 0 \text{ etc.}]$$

Hence, $OA = OB = OC$, where O is the origin and A, B, C are the points representing z_1, z_2 and z_3 , respectively. Therefore, O is circumcenter of $\triangle ABC$. Now,



$$\arg\left(\frac{z_3}{z_2}\right) = \angle BOC \quad (1)$$

$$= 2\angle BAC = 2 \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) \quad (2)$$

$$= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2 \quad [\because \angle BOC = 2\angle BAC]$$

$$\text{Hence, } \arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2$$

Also, centroid is $(z_1 + z_2 + z_3)/3$. Since $HG:GO \equiv 2:1$ (where H is orthocenter and G is centroid), then orthocenter is $z_1 + z_2 + z_3$ (by section formula). When triangle is equilateral centroid coincides with circumcenter; hence $z_1 + z_2 + z_3 = 0$.

Also, the area for equilateral triangle is $(\sqrt{3}/4)L^2$, where L is length of side. Since radius is $|z_1|$, $L = \sqrt{3}|z_1|$, the area is $(3\sqrt{3}/4)|z_1|^2$.

27. (2), (3)

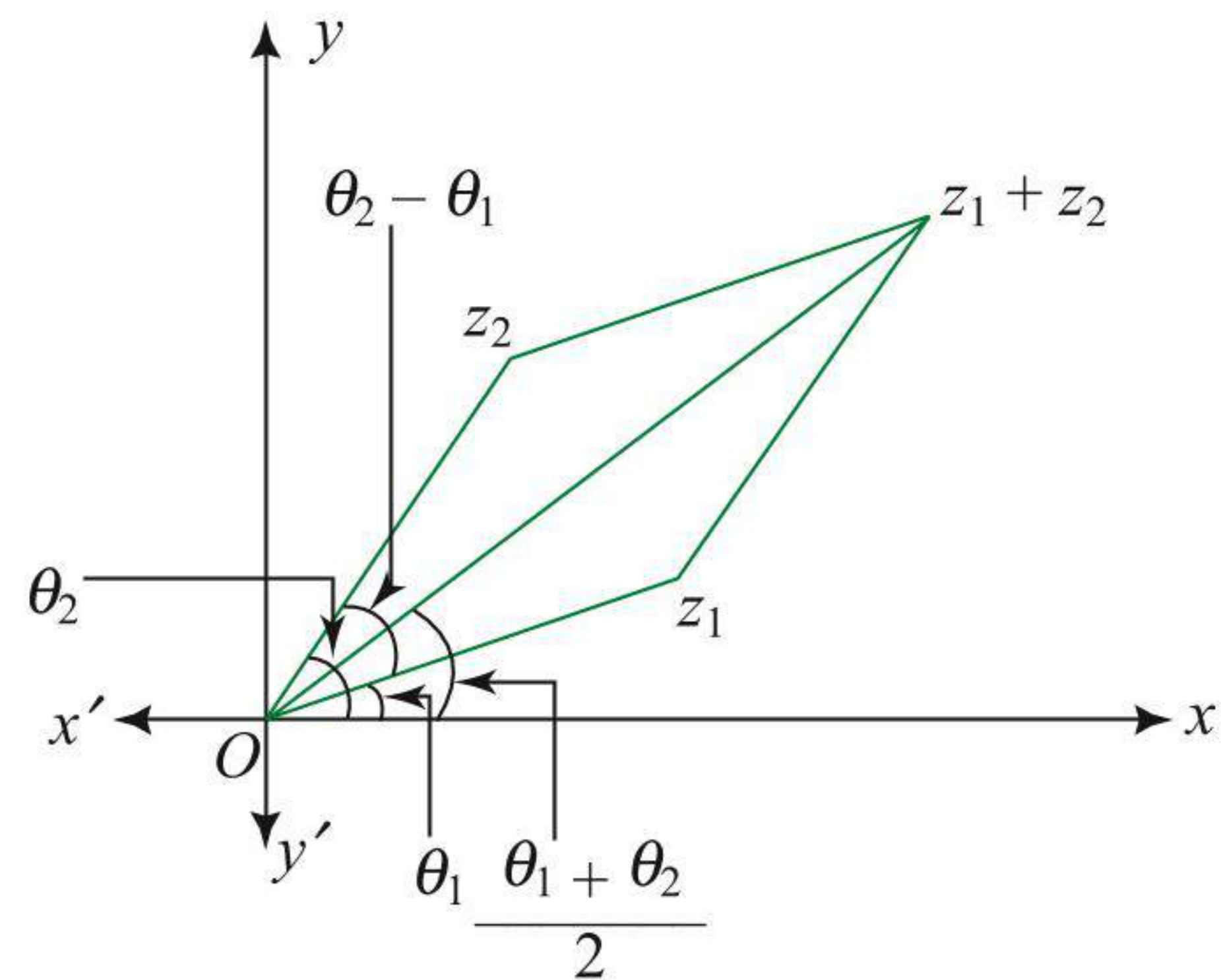
z_1 and z_2 are the roots of the equation $z^2 - az + b = 0$. Hence,

$$z_1 + z_2 = a, z_1 z_2 = b$$

Now, $|z_1 + z_2| \leq |z_1| + |z_2|$

$$\Rightarrow |z_1 + z_2| = |a| \leq 1 + 1 = 2 \quad (\because |z_1| = |z_2| = 1)$$

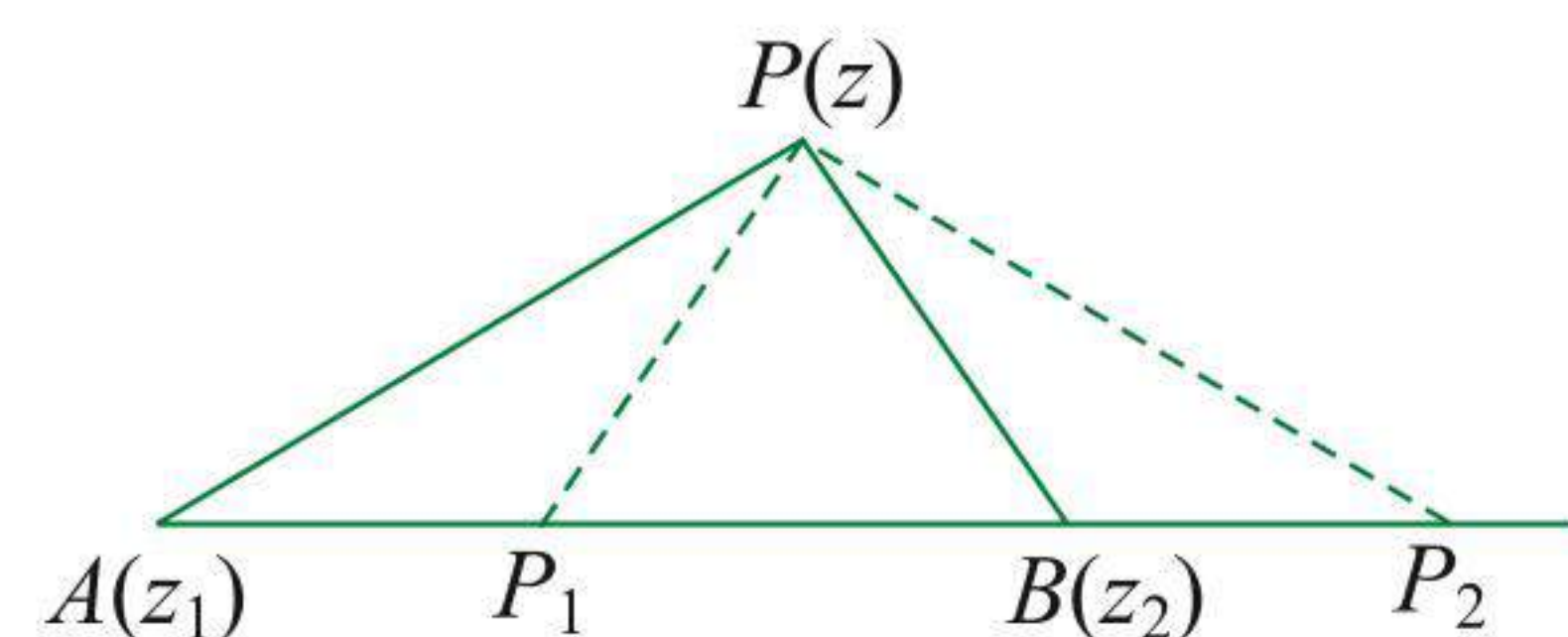
$$\Rightarrow \arg(a) = \frac{1}{2}[\arg(z_2) + \arg(z_1)]$$



Also, $\arg(b) = \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

$$\Rightarrow 2 \arg(a) = \arg(b)$$

28. (2), (3)



Let internal and external bisectors of $\angle APB$ meet the line joining A and B at P_1 and P_2 , respectively. Hence,

$$AP_1 : P_1 B \equiv PA : PB \equiv 3 : 1 \text{ (internal division)}$$

$$AP_2 : P_2 B \equiv PA : PB \equiv 3 : 1 \text{ (external division)}$$

Thus, P_1 and P_2 are fixed points. Also,

$$\angle P_1 P P_2 = \frac{\pi}{2}$$

Thus, P lies on a circle having $P_1 P_2$ as its diameter. Clearly, $B(z_2)$ lies inside this circle.

29. (1), (2), (4)

$$\left| \frac{2z - i}{z + 1} \right| = m$$

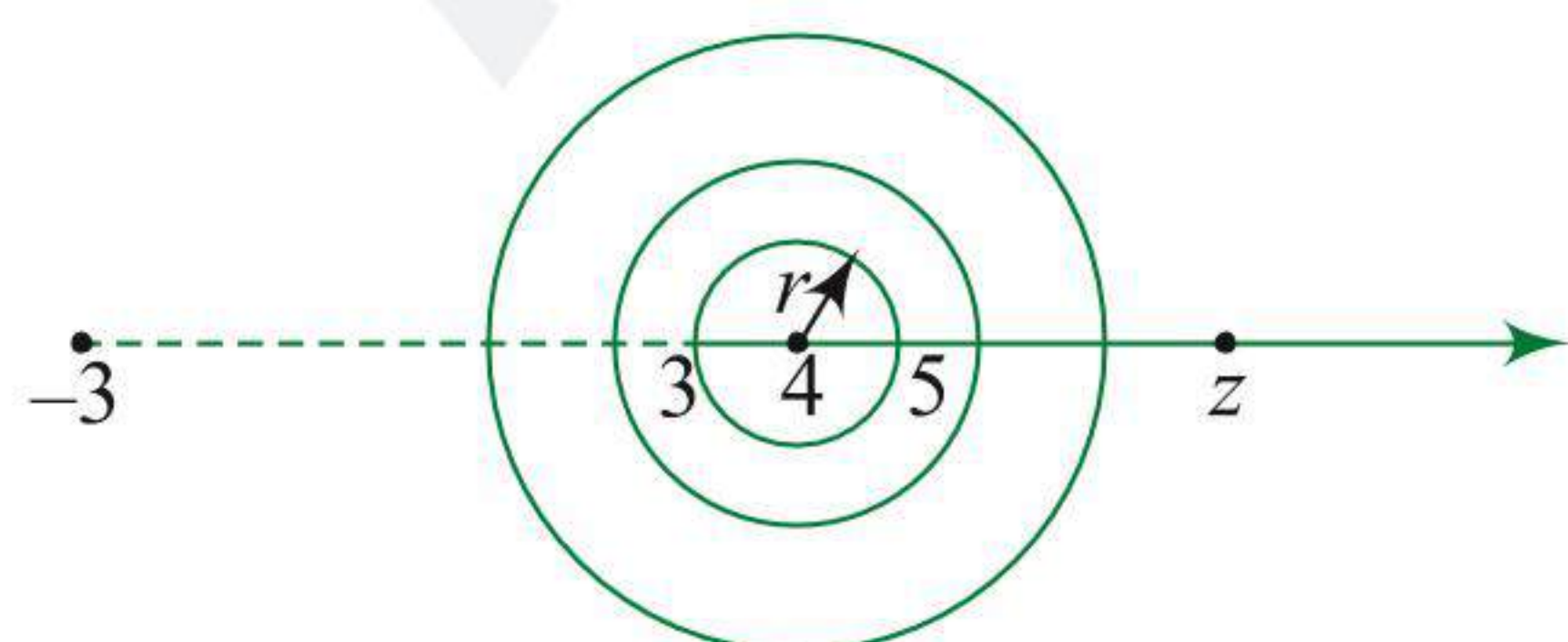
$$\Rightarrow \left| z - \frac{i}{2} \right| = \frac{m}{2} |z + 1|$$

This shows that the given equation will represent a circle, if $m/2 \neq 0, 1$, i.e., $m \neq 0, 2$.

30. (1), (3), (4)

Equation $|z + 3| - |z - 3| = 6$ represents a ray to the right of complex number '3'.

$|z - 4| = r$ represents a circle whose center is 4 and radius r .



When radius of circle $r = 1$, it cuts ray at two points '3' and '5'.

For radius $r < 1$, circle cuts ray at two points.

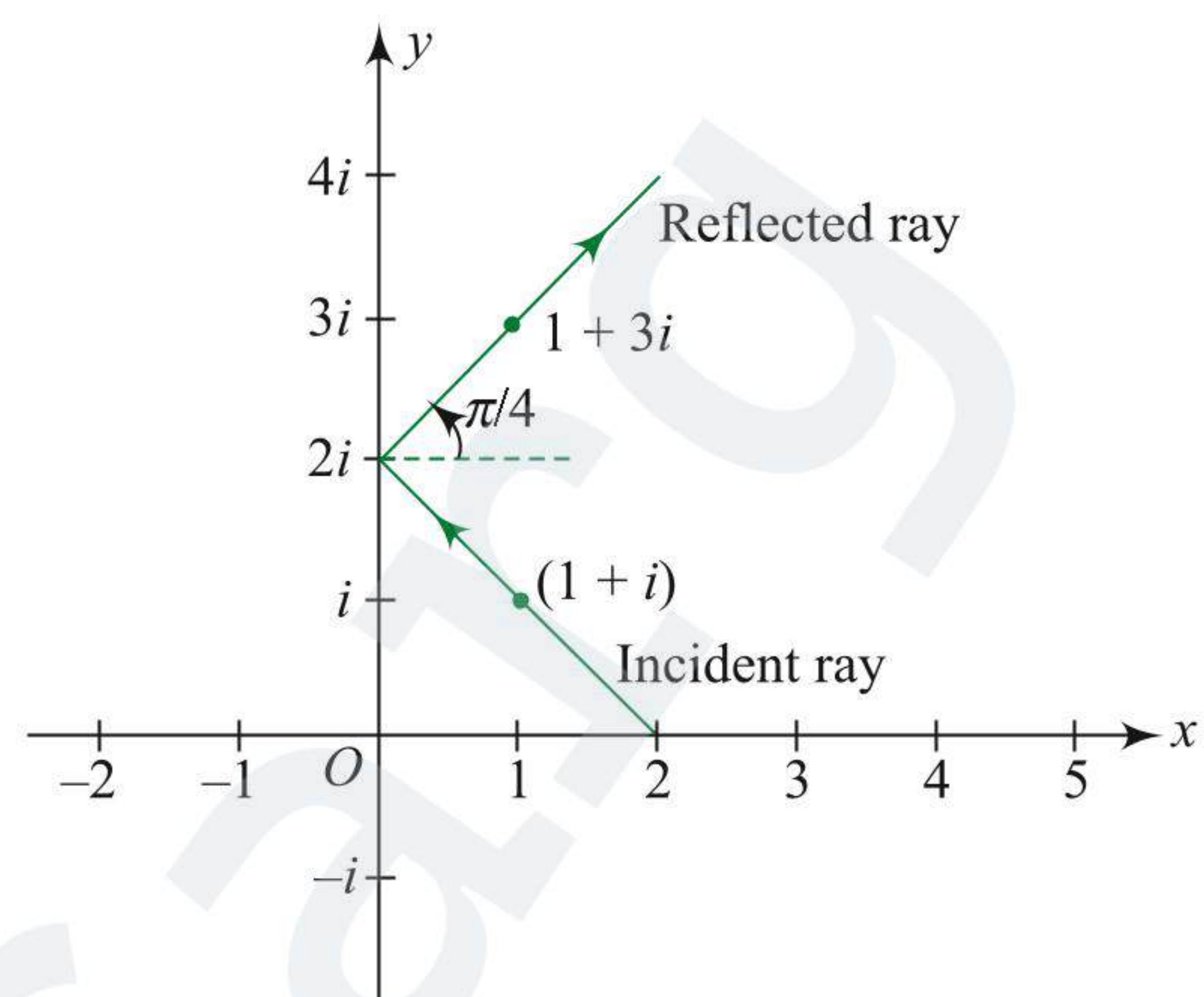
When radius of circle $r > 1$, it cuts ray at only one point.

31. (1), (2)

Incident ray is $|z - 2| - |z - 1 - i| = \sqrt{2} = |2 - (1 + i)|$

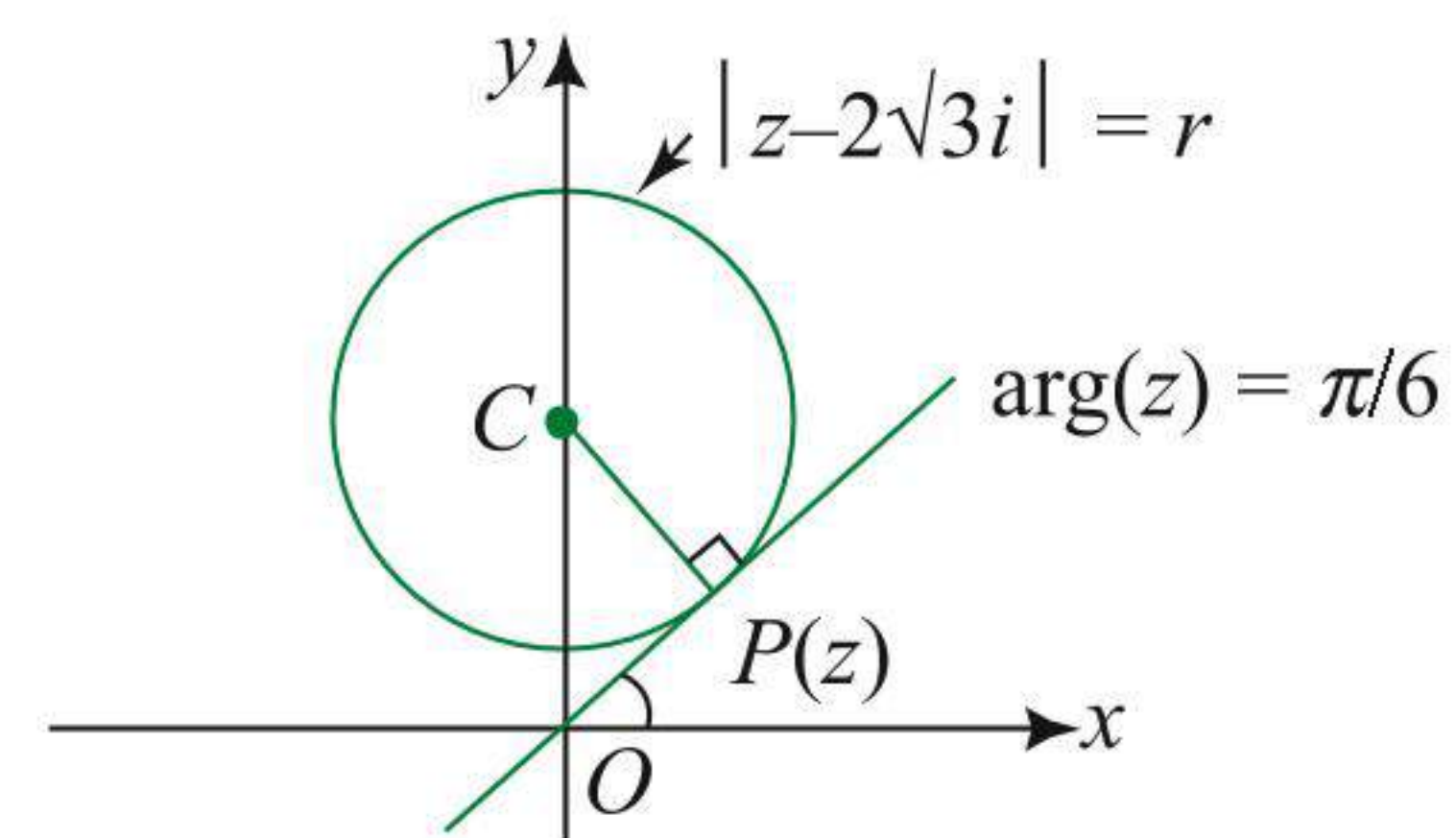
This ray is emanating from the point '2' and moving to the left of '2'.

It strikes y -axis at point $2i$.



Then equation of reflected ray is $\arg(z - 2i) = \frac{\pi}{4}$
or $|z - 2i| - |z - 3 - i| = \sqrt{2}$.

32. (1), (4)



$$CP = r, OC = 2\sqrt{3}, \angle COP = \pi/3$$

$$\Rightarrow CP = OC \sin \frac{\pi}{3} = 2\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 3$$

Thus, when $r = 3$, the circle touches the line. Hence, for two distinct points of intersection, $3 < r < 2\sqrt{3}$.

33. (1), (2), (4)

We have $z_1 + 2 = 3z$ and $-z_2 - 2 = 3z$

$$\Rightarrow |z_1 + 2| = 3|z| = 3$$

$$\text{and } z_2 + 2 = -3z$$

$$\Rightarrow |z_2 + 2| = |-3z| = 3$$

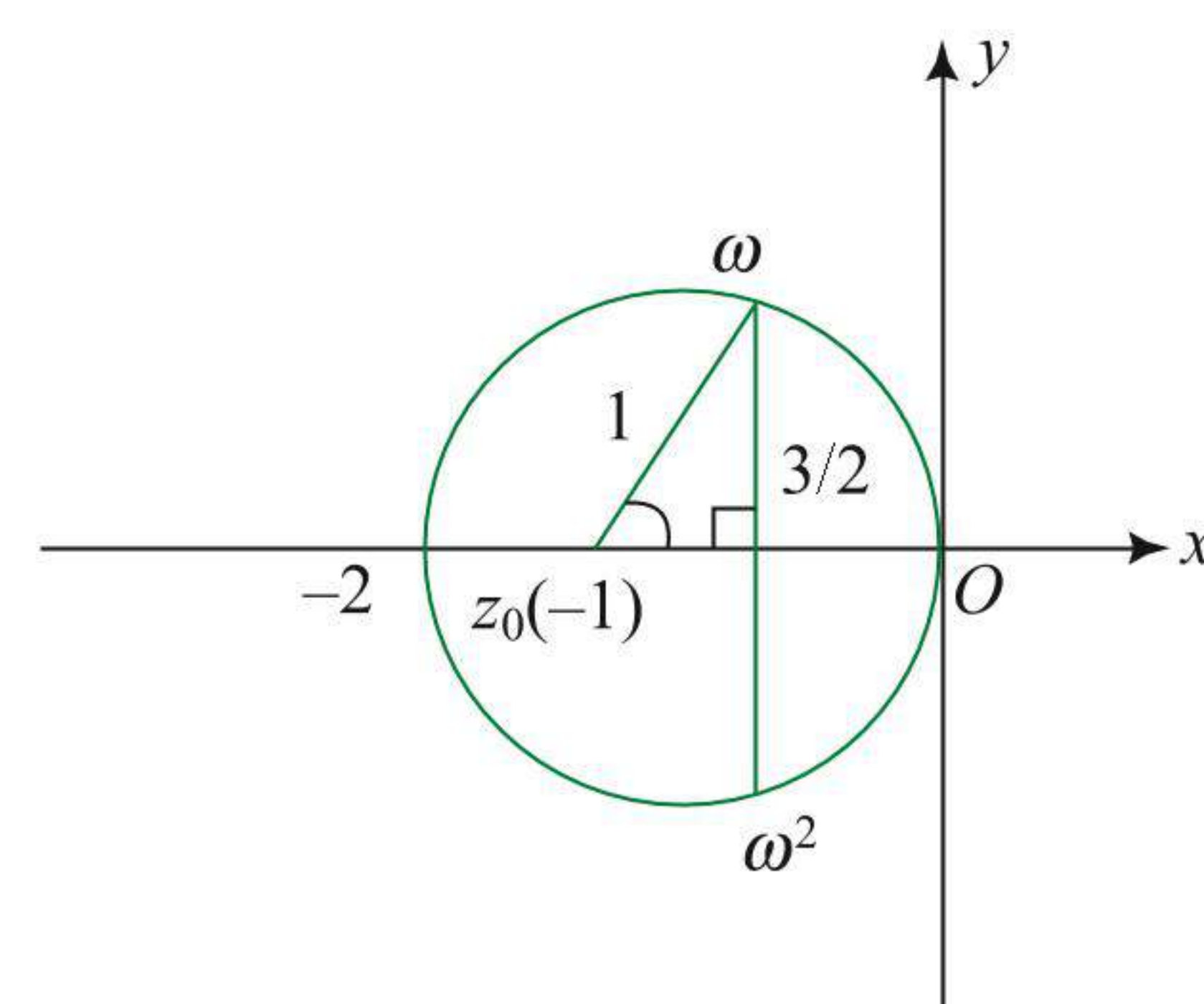
Thus, z_1 and z_2 describe the same circle having center at '-2' and radius '3'.

Now, $z_1 - z_2 = 6z$

$$\Rightarrow |z_1 - z_2| = 6|z| = 6$$

So, $(z_1 - z_2)$ moves on a circle with centre as origin and radius 6.

34. (1), (4)



$$\text{Given } |z_0| = |z_0 - \omega| = |z_0 - \omega^2|$$

So, the distance of z_0 from 0, ω and ω^2 is same.

So, z_0 is circumcenter of triangle joining ω and ω^2 .

$$\therefore z_0 = -1$$

Hence, set $S = \{z : x = x + iy, y \geq 0, |z + 1| \leq 1\}$

Therefore, minimum value of $|z|$ is 0.

Also, $\arg(\omega - z_0)$ = angle made by line joining $z_0 (= -1)$ to ω

$$\text{with real axis} = \frac{\pi}{3}$$

35. (2), (3)

We have

$$\left| \frac{1}{z_2} + \frac{1}{z_1} \right| = \left| \frac{1}{z_2} - \frac{1}{z_1} \right|$$

$$\Rightarrow |z_1 + z_2| = |z_1 - z_2|$$

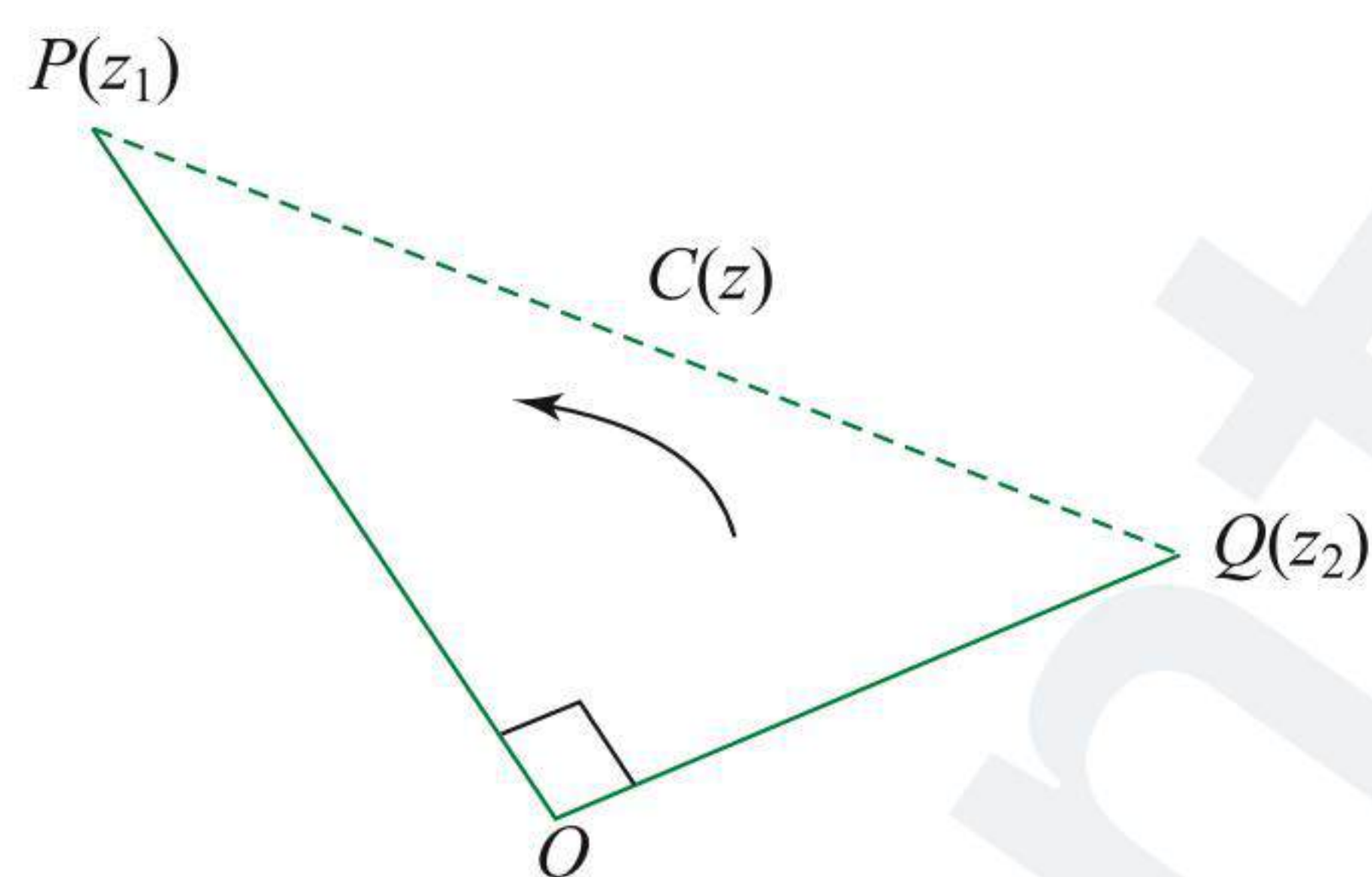
Squaring both sides, we have

$$\begin{aligned} |z_1|^2 + |z_2|^2 + 2(z_1 \bar{z}_2 + \bar{z}_1 z_2) \\ = |z_1|^2 + |z_2|^2 - 2(z_1 \bar{z}_2 + \bar{z}_1 z_2) \end{aligned}$$

$$\Rightarrow 4(z_1 \bar{z}_2 + \bar{z}_1 z_2) = 0$$

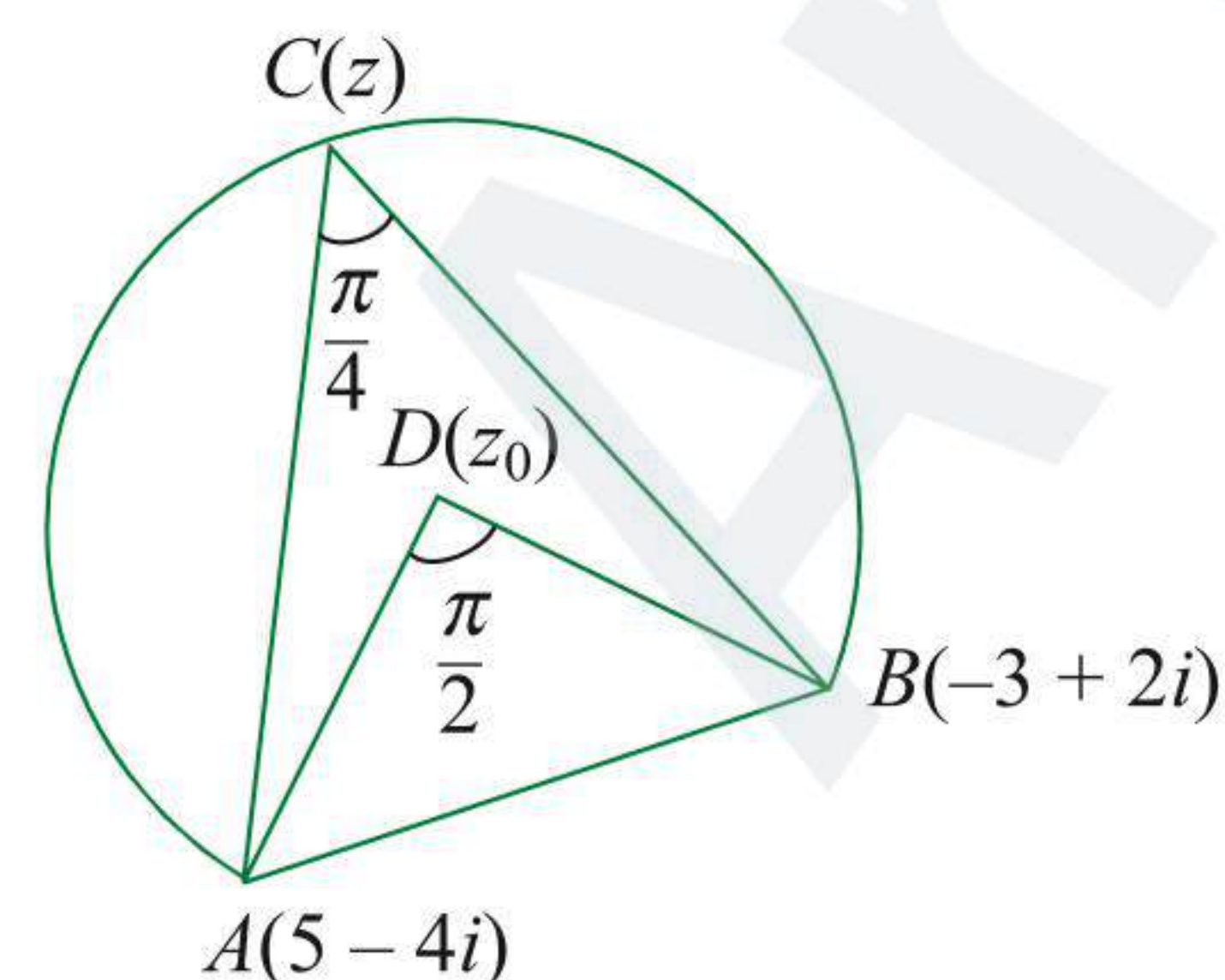
$$\Rightarrow \frac{z_1}{z_2} + \frac{\bar{z}_1}{\bar{z}_2} = 0$$

$$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} = \arg\left(\frac{z_1 - 0}{z_2 - 0}\right)$$



The angle between z_2 , O , and z_1 is a right angle, taken in order, as shown in the above diagram. Now, the circumcenter of the above diagram will lie on the line PQ as diameter and is represented by C which is the center of PQ , such that $z = (z_1 + z_2)/2$, where z is the affix of circumcenter.

36. (1), (3), (4)



$$\frac{z_0 - (-3 + 2i)}{z_0 - (5 - 4i)} = \frac{BD}{AD} e^{i\pi/2} = i$$

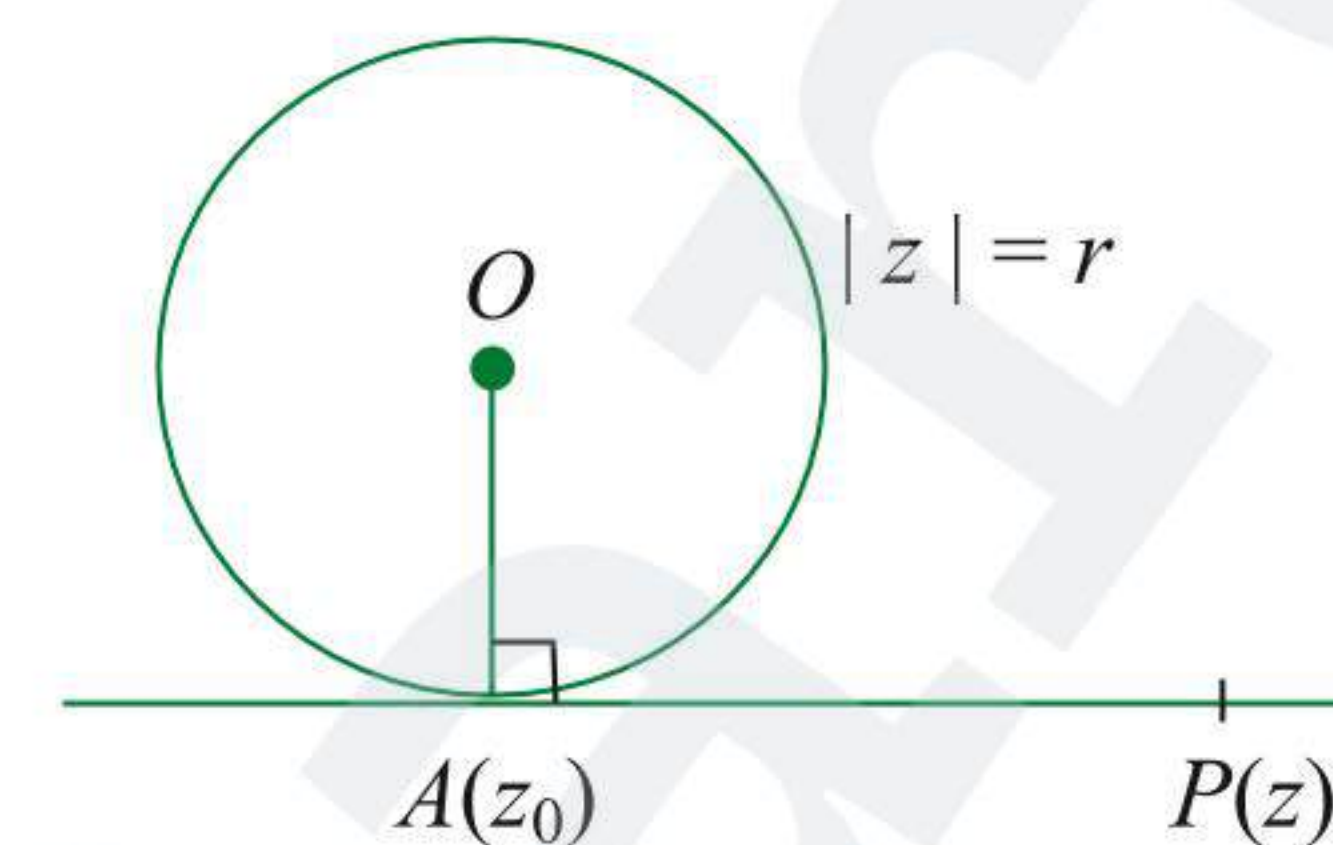
$$\Rightarrow z_0 + 3 - 2i = iz_0 - 5i - 4$$

$$\Rightarrow z_0 = -2 - 5i$$

$$\begin{aligned} \Rightarrow \text{Radius } AD &= |5 - 4i - (-2 - 5i)| \\ &= |7 + i| \\ &= \sqrt{50} = 5\sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{Length of arc} &= \frac{3}{4} (\text{Perimeter of circle}) \\ &= \frac{3}{4} (2\pi \times 5\sqrt{2}) \\ &= \frac{15\pi}{\sqrt{2}} \end{aligned}$$

37. (1), (2)



$$\angle OAP = \frac{\pi}{2}$$

$$\Rightarrow \frac{z - z_0}{z_0} \text{ is purely imaginary}$$

$$\Rightarrow \frac{z - z_0}{z_0} + \frac{\bar{z} - \bar{z}_0}{\bar{z}_0} = 0$$

$$\text{or } \frac{z}{z_0} + \frac{\bar{z}}{\bar{z}_0} = 2 \quad (1)$$

$$\Rightarrow \operatorname{Re}\left(\frac{z}{z_0}\right) = 1$$

From (1),

$$z\bar{z}_0 + z_0\bar{z} = 2|z_0|^2 = 2r^2$$

38. (1), (3)

Given,

$$z^n = (z + 1)^n \Rightarrow |z^n| = |(z + 1)^n|$$

$$\therefore |z|^n = |z + 1|^n \Rightarrow |z| = |z + 1|$$

$$\Rightarrow |z|^2 = |z + 1|^2$$

$$\Rightarrow x^2 + y^2 = (x + 1)^2 + y^2, \text{ where } z = x + iy$$

$$\Rightarrow x = -\frac{1}{2}$$

Hence, z lies on the line $x = -1/2$. Hence, sum of real parts of the roots is $-(n-1)/2$ (since equation has $n-1$ roots).

39. (1), (2)

$$\left| z - \frac{1}{z} \right| = 1$$

$$\Rightarrow 1 \geq \left| z - \frac{1}{z} \right|$$

$$\Rightarrow -1 \leq z - \frac{1}{z} \leq 1$$

$$\Rightarrow -|z| \leq |z|^2 - 1 \leq |z|$$

From $|z|^2 - 1 \geq -|z|$, we get

$$|z|^2 + |z| - 1 \geq 0$$

$$\Rightarrow |z| \geq \frac{-1 + \sqrt{5}}{2} \quad (1)$$

From $|z|^2 - 1 \leq |z|$, we get

$$|z|^2 - |z| - 1 \leq 0$$

$$\Rightarrow \frac{1-\sqrt{5}}{2} \leq |z| \leq \frac{1+\sqrt{5}}{2} \quad (2)$$

From (1) and (2), we get

$$\Rightarrow \frac{-1+\sqrt{5}}{2} \leq |z| \leq \frac{1+\sqrt{5}}{2}$$

$$\Rightarrow |z|_{\min} = \frac{\sqrt{5}-1}{2}, |z|_{\max} = \frac{1+\sqrt{5}}{2}$$

40. (1), (2), (4)

$$x^n - 1 = (x-1)(x-z_1)(x-z_2) \cdots (x-z_{n-1})$$

$$\Rightarrow \frac{x^n - 1}{x - 1} = (x - z_1)(x - z_2) \cdots (x - z_{n-1})$$

Putting $x = \omega$, we have

$$\prod_{r=1}^{n-1} (\omega - z_r) = \frac{\omega^n - 1}{\omega - 1} = \begin{cases} 0, & \text{if } n=3k, k \in \mathbb{Z} \\ 1, & \text{if } n=3k+1, k \in \mathbb{Z} \\ 1+\omega, & \text{if } n=3k+2, k \in \mathbb{Z} \end{cases}$$

41. (1), (3)

If $p = q$, then equation becomes $z^p = \bar{z}^q$ and it has infinite number of solutions because any $z \in \mathbb{R}$ will satisfy it. If $p \neq q$, let $p > q$, then $z^p = \bar{z}^q$.

$$\therefore |z|^p = |z|^q$$

$$\Rightarrow |z|^p (|z|^{p-q} - 1) = 0$$

$$\Rightarrow |z| = 0 \text{ or } |z| = 1$$

$$|z| = 0 \Rightarrow z = 0 + i0$$

$$|z| = 1 \Rightarrow z = e^{i\theta}$$

$$\Rightarrow e^{(p+q)\theta i} = 1$$

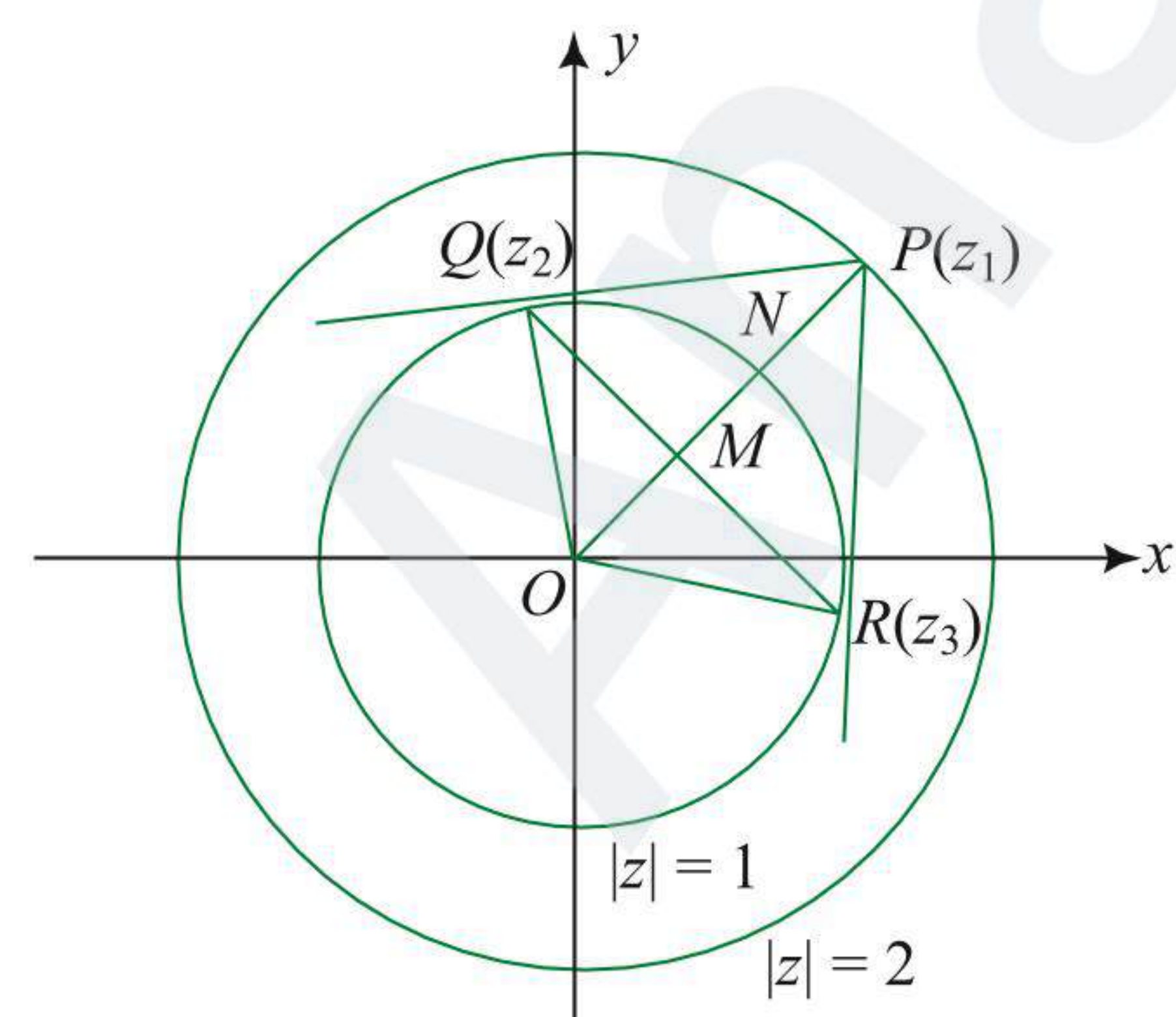
$$\Rightarrow z = 1^{1/(p+q)}$$

Therefore, the number of solutions is $p + q + 1$.

42. (1), (2), (3), (4)

Number of roots common to $z^{n_1} = 1$ and $z^{n_2} = 1$ is HCF (n_1, n_2)

43. (1), (2), (3), (4)



Since $OQ = 1$ and $OP = 2$, so $\sin(\angle OPQ) = 1/2$ and hence $\angle QPR = \pi/3$. Then $\triangle PQR$ is equilateral. Also, $OM \perp QR$. Then from $\triangle OMQ$, $OM = 1/2$. Hence, $MN = 1/2$. Then centroid of $\triangle PQR$ lies on $|z| = 1$.

As PQR is an equilateral triangle, so orthocenter, circumcenter, and centroid will coincide. Now,

$$\left| \frac{z_1 + z_2 + z_3}{3} \right| = 1$$

$$\text{or } |z_1 + z_2 + z_3|^2 = 9$$

$$\text{or } (z_1 + z_2 + z_3)(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) = 9$$

$$\text{or } \left(\frac{4}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) \left(\frac{4}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} \right) = 9$$

and $\angle QOR = 120^\circ$

44. (1), (3), (4)

$$z' = ze^{i\alpha} \quad (1)$$

$$z'' = ze^{-i\alpha} \quad (2)$$

$$\therefore z'z'' = z^2$$

$$\Rightarrow z', z, z'' \text{ are in G.P.}$$

$$\text{Also, } \left(\frac{z'}{z} \right)^2 + \left(\frac{z''}{z} \right)^2 = 2 \cos 2\alpha$$

$$\Rightarrow z'^2 + z''^2 = 2z^2 \cos 2\alpha$$

$$= 2z^2 (2\cos^2 \alpha - 1)$$

$$\text{or } z'^2 + z''^2 + 2z^2 = 4z^2 \cos^2 \alpha$$

$$\text{or } z'^2 + z''^2 + 2z'z'' = 4z^2 \cos^2 \alpha$$

$$\text{or } (z' + z'')^2 = 4z^2 \cos^2 \alpha$$

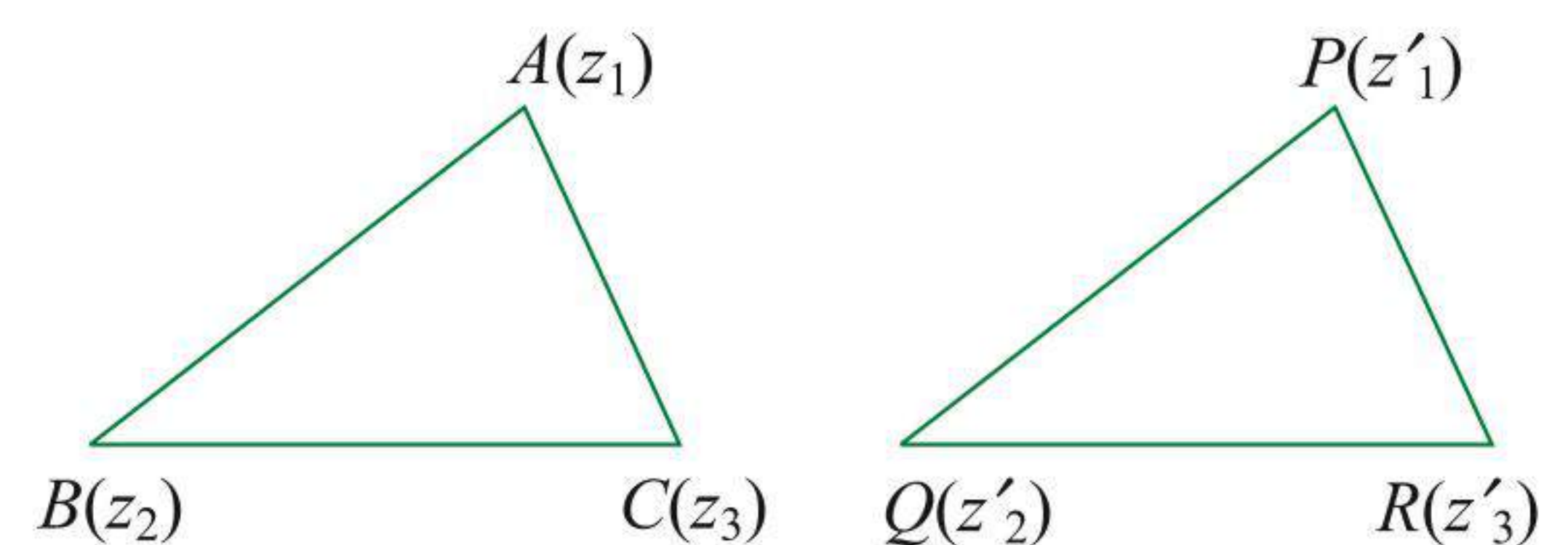
$$\text{or } z' + z'' = 2z \cos \alpha$$

45. (1), (2), (3), (4)

$$z_3 = (1 - \lambda)z_1 + \lambda z_2 = \frac{(1 - \lambda)z_1 + \lambda z_2}{1 - \lambda + \lambda}$$

Hence, z_3 divides the line joining $A(z_1)$ and $B(z_2)$ in the ratio $\lambda:(1 - \lambda)$. That means the given points are collinear. Also, the ratio $\lambda/(1 - \lambda) > 0$ (or $0 < \lambda < 1$) if z_3 divides the line joining z_1 and z_2 internally and $\mu/(1 - \mu) < 0$ (or $\mu < 0$ or $\mu > 1$) if z'_3 divides the line joining z'_1, z'_2 externally.

When λ, μ are complex numbers, where $\lambda = \mu$, we have $z_3 = (1 - \lambda)z_1 + \lambda z_2$ and $z'_3 = (1 - \lambda)z'_1 + \lambda z'_2$.



Comparing the value of λ , we have

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{z'_3 - z'_1}{z'_2 - z'_1}$$

$$\Rightarrow \left| \frac{z_3 - z_1}{z_2 - z_1} \right| = \left| \frac{z'_3 - z'_1}{z'_2 - z'_1} \right| \text{ and } \arg \left(\frac{z_3 - z_1}{z_2 - z_1} \right) = \arg \left(\frac{z'_3 - z'_1}{z'_2 - z'_1} \right)$$

$$\Rightarrow \frac{AC}{AB} = \frac{PR}{PQ} \text{ and } \angle BAC = \angle QPR$$

Hence, triangles ABC and PQR are similar.

46. (1), (3), (4)

Choice (a) on simplification gives

$$z = \frac{1+x}{1+x^2} + i \frac{1+x}{1+x^2}$$

For $x = 0.5$, $f(0.5) > 1$ which is out of range, Hence, (1) is not correct. From choice (2),

$$z = \frac{1-x}{1+x^2} + i \frac{1-x}{1+x^2}$$

$f(x)$ and $g(x) \in (0, 1)$ if $x \in (0, 1)$. Hence, (2) is correct. From choice (3),

$$z = \frac{1+x}{1+x^2} + \frac{1-x}{1+x^2}i$$

Hence, (3) is not correct. From choice (4),

$$z = \frac{1-x}{1+x^2} + \frac{1+x}{1+x^2}i$$

Hence, (4) is not correct.

47. (1), (2), (4)

It is given that z_1 and z_2 are the roots of the equation $az^2 + bz + c = 0$.

$$\text{So, } |z_1 + z_2| = \left| -\frac{b}{a} \right| = 1$$

$$\text{And } |z_1 z_2| = \left| \frac{c}{a} \right| = 1$$

$$\begin{aligned} \therefore |z_2| &= 1 \\ |z_1 + z_2|^2 &= 1 \\ \therefore 2 + z_1 \bar{z}_2 + z_2 \bar{z}_1 &= 1 \end{aligned}$$

$$\begin{aligned} \text{Now, } z_2 &= z_1 e^{i\theta} \\ \therefore |z_1 + z_1 e^{i\theta}| &= |z_1| |1 + e^{i\theta}| = 1 \end{aligned}$$

$$\therefore 2 \cos \frac{\theta}{2} = 1$$

$$\therefore \theta = \frac{2\pi}{3}$$

$$\text{Now, } \frac{(z_1 + z_2)^2}{z_1 z_2} = 1$$

$$\Rightarrow \frac{b^2}{a^2} = \frac{c}{a}$$

$$\begin{aligned} \Rightarrow b^2 &= ac \\ PQ &= |z_1 - z_2| = \sqrt{3} \end{aligned}$$

48. (1), (2), (3), (4)

Let the roots of $az^3 + bz^2 + cz + d = 0$ be $z_1 = x_1$, $z_2 = x_2 + iy_2$ and $z_3 = x_2 - iy_2$.

$$\therefore z_1 + z_2 + z_3 = -\frac{b}{a}$$

$$\Rightarrow x_1 + 2x_2 = -\frac{b}{a} < 0$$

$$\Rightarrow ab > 0$$

$$z_1 z_2 + z_2 z_3 + z_1 z_3 = \frac{c}{a}$$

$$\begin{aligned} x_1(x_2 + iy_2) + (x_2^2 + y_2^2) + x_1(x_2 - iy_2) \\ = 2x_1 x_2 + x_2^2 + y_2^2 > 0 \end{aligned}$$

$$\Rightarrow \frac{c}{a} > 0$$

$$\Rightarrow ac > 0$$

$$\Rightarrow a^2 bc > 0$$

$$\Rightarrow bc > 0$$

$$\text{Also, } z_1 z_2 z_3 = x_1(x_2^2 + y_2^2) = -\frac{d}{a}$$

$$\Rightarrow ad > 0$$

$$\begin{aligned} \text{Further, } -\frac{bc}{a^2} &= \left(-\frac{b}{a}\right)\left(\frac{c}{a}\right) \\ &= (x_1 + 2x_2)(2x_1 x_2 + x_2^2 + y_2^2) \\ &= x_1(x_2^2 + y_2^2) + 2x_1^2 x_2 + 2x_1 x_2^2 + 2x_2(x_2^2 + y_2^2) \\ &< x_1(x_2^2 + y_2^2) \end{aligned}$$

$$\Rightarrow -\frac{bc}{a^2} < -\frac{d}{a}$$

$$\Rightarrow bc > ad$$

49. (1), (2), (3)

$$\frac{3}{2 + e^{i\theta}} = ax + iby$$

$$\Rightarrow \frac{3}{ax + iby} - 2 = e^{i\theta}$$

$$\Rightarrow \frac{3 - 2ax - 2iby}{ax + iby} = e^{i\theta}$$

$$\Rightarrow \left| \frac{3 - 2ax - 2iby}{ax + iby} \right| = |e^{i\theta}| = 1$$

$$\Rightarrow (3 - 2ax)^2 + 4b^2 y^2 = a^2 x^2 + b^2 y^2$$

When $a = 1$ and $b = 2$, we have

$$(3 - 2x)^2 + 16y^2 = x^2 + 4y^2$$

$$\Rightarrow 3x^2 + 12y^2 - 12x + 9 = 0$$

$$\Rightarrow x^2 - 4x + 4y^2 + 3 = 0; \text{ which is ellipse}$$

When $a = b = 1$, we have

$$(3 - 2x)^2 + 4y^2 = x^2 + y^2$$

$$\Rightarrow 3x^2 + 3y^2 - 12x + 9 = 0$$

$$\text{or } x^2 + y^2 - 4x + 3 = 0; \text{ which is circle}$$

When $a = 1$ and $b = 0$, we have

$$(3 - 2x)^2 = x^2; \text{ which is pair of straight lines}$$

Linked Comprehension Type

1. (1), 2. (4), 3. (3), 4. (4)

$$\begin{aligned} z &= \frac{1 - i \sin \theta}{1 + i \cos \theta} = \frac{(1 - i \sin \theta)(1 - i \cos \theta)}{(1 + i \cos \theta)(1 - i \cos \theta)} \\ &= \frac{(1 - \sin \theta \cos \theta) - i(\cos \theta + \sin \theta)}{(1 + \cos^2 \theta)} \end{aligned}$$

If z is purely real, then

$$\cos \theta + \sin \theta = 0$$

$$\text{or } \tan \theta = -1$$

$$\Rightarrow \theta = n\pi - \frac{\pi}{4}, n \in I$$

If z is purely imaginary, $1 - \sin \theta \cos \theta = 0$ or $\sin \theta \cos \theta = 1$, which is not possible.

$$|z| = \left| \frac{1 - i \sin \theta}{1 + i \cos \theta} \right| = \frac{\sqrt{1 + \sin^2 \theta}}{\sqrt{1 + \cos^2 \theta}}$$

If $|z| = 1$, then

$$\cos^2 \theta = \sin^2 \theta$$

$$\Rightarrow \tan^2 \theta = 1$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{4}, n \in I$$

We have,

$$\arg(z) = \tan^{-1} \left(\frac{-(\cos \theta + \sin \theta)}{(1 - \sin \theta \cos \theta)} \right)$$

Now, $\arg(z) = \pi/4$

$$\begin{aligned} \Rightarrow \frac{-(\cos\theta + \sin\theta)}{(1 - \sin\theta \cos\theta)} &= 1 \\ \Rightarrow \cos^2\theta + \sin^2\theta + 2\sin\theta \cos\theta &= 1 + \sin^2\theta \cos^2\theta - 2\sin\theta \cos\theta \\ \Rightarrow 1 + 4\sin\theta \cos\theta &= 1 + \sin^2\theta \cos^2\theta \\ \Rightarrow \sin^2\theta \cos^2\theta - 4\sin\theta \cos\theta &= 0 \\ \Rightarrow \sin\theta \cos\theta (\sin\theta \cos\theta - 4) &= 0 \\ \Rightarrow \sin\theta \cos\theta = 0 \quad (\because \sin\theta \cos\theta = 4 \text{ is not possible}) \\ \Rightarrow \theta = (2n+1)\pi \text{ or } \theta = (4n-1)\pi/2, n \in I \\ &(\because -\cos\theta - \sin\theta > 0) \end{aligned}$$

5. (2), 6. (2), 7. (3), 8. (3)

Given that

$$\begin{aligned} |z_1 + z_2|^2 &= |z_1|^2 + |z_2|^2 \\ \Rightarrow |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1z_2 &= |z_1|^2 + |z_2|^2 \\ \Rightarrow z_1\bar{z}_2 + \bar{z}_1z_2 &= 0 \quad (1) \\ \Rightarrow \frac{z_1}{z_2} + \frac{\bar{z}_1}{\bar{z}_2} &= 0 \quad (\text{dividing by } z_2\bar{z}_2) \\ \Rightarrow \frac{z_1}{z_2} + \overline{\left(\frac{z_1}{z_2}\right)} &= 0 \quad (2) \end{aligned}$$

From (1), $z_2\bar{z}_2$ is purely imaginary. From (2), z_1/z_2 is purely imaginary. Hence,

$$\arg\left(\frac{z_1}{z_2}\right) = \pm \frac{\pi}{2} \text{ or } \arg(z_1) - \arg(z_2) = \pm \frac{\pi}{2}$$

Also, $i(z_1/z_2)$ is purely real. Hence its possible arguments are 0 and π .

9. (1), 10. (2), 11. (3)

$$z = -\lambda \pm \sqrt{\lambda^2 - 1}$$

Case I:

When $-1 < \lambda < 1$, we have

$$\lambda^2 < 1 \Rightarrow \lambda^2 - 1 < 0$$

$$z = -\lambda \pm i\sqrt{1 - \lambda^2}$$

$$\begin{aligned} \text{or } x &= -\lambda, y = \pm i\sqrt{1 - \lambda^2} \\ \Rightarrow y^2 &= 1 - x^2 \text{ or } x^2 + y^2 = 1 \end{aligned}$$

Case II:

$$\lambda > 1 \Rightarrow \lambda^2 - 1 > 0$$

$$z = -\lambda \pm \sqrt{\lambda^2 - 1}$$

$$\text{or } x = -\lambda \pm \sqrt{\lambda^2 - 1}, y = 0$$

Roots are $(-\lambda + \sqrt{\lambda^2 - 1}, 0)$, $(-\lambda - \sqrt{\lambda^2 - 1}, 0)$. One root lies inside the unit circle and the other root lies outside the unit circle.

Case III:

When λ is very large, then

$$z = -\lambda - \sqrt{\lambda^2 - 1} \approx -2\lambda$$

$$\begin{aligned} z &= -\lambda + \sqrt{\lambda^2 - 1} = \frac{(-\lambda + \sqrt{\lambda^2 - 1})(-\lambda - \sqrt{\lambda^2 - 1})}{(-\lambda - \sqrt{\lambda^2 - 1})} \\ &= \frac{1}{-\lambda - \sqrt{\lambda^2 - 1}} = -\frac{1}{2\lambda} \end{aligned}$$

12. (2), 13. (3)

Given equation is $z(z^3 + az^2 + (12 + 9i)z + b) = 0$

So, either $z = 0$

$$\text{or } z^3 + az^2 + (12 + 9i)z + b = 0 \quad \dots(1)$$

So, one of the vertices of the square is origin.

Therefore, other three vertices are z_1 , iz_1 and $(z_1 + iz_1)$ which are the roots equation (1).

$$\begin{aligned} \therefore z_1(iz_1) + (iz_1)(z_1 + iz_1) + z_1(z_1 + iz_1) &= 12 + 9i \\ \Rightarrow z_1^2(i + i + i^2 + 1 + i) &= 12 + 9i \\ \Rightarrow 3iz_1^2 &= 12 + 9i \\ \Rightarrow z_1^2 &= 3 - 4i \\ \Rightarrow z_1 &= \sqrt{3 - 4i} = \pm(2 - i) \\ \therefore z_1 &= 2 - i \text{ or } -2 + i \text{ (both values will give the same result)} \\ \text{Also, } -a = z_1 + iz_1 + z_1(1 + i) &= 2z_1(1 + i) \\ &= 2(2 - i)(1 + i) = 6 + 2i \\ \therefore a &= -6 - 2i \\ \text{Further, } (z_1)(iz_1)(1 + i)z_1 &= -b \\ \therefore -b &= z_1^3(-1 + i) \\ \text{or } b &= (2 - i)^3(1 - i) = -9 - 13i \\ b &= -9 - 13i \\ |a - b| &= |3 + 11i| = \sqrt{130} \\ |z_1| &= \sqrt{5} \end{aligned}$$

Therefore, area of square is 5 sq. units.

14. (1), 15. (4), 16. (4).

Let z_1 (purely imaginary) be a root of the given equation. Then,

$$z_1 = -\bar{z}_1 \text{ and } az_1^2 + bz_1 + c = 0 \quad (1)$$

$$\Rightarrow \overline{az_1^2 + bz_1 + c} = 0$$

$$\Rightarrow \bar{a}\bar{z}_1^2 + \bar{b}\bar{z}_1 + \bar{c} = 0$$

$$\Rightarrow \bar{a}z_1^2 - \bar{b}z_1 + \bar{c} = 0 \quad (\text{as } \bar{z}_1 = -z_1) \quad (2)$$

Now Eqs. (1) and (2) must have one common root.

$$\therefore (c\bar{a} - a\bar{c})^2 = (b\bar{c} + c\bar{b})(-a\bar{b} - \bar{a}b)$$

Let z_1 and z_2 be two purely imaginary roots. Then,

$$\bar{z}_1 = -z_1, \bar{z}_2 = -z_2$$

$$\text{Now, } az^2 + bz + c = 0 \quad (3)$$

$$\text{or } \overline{az^2 + bz + c} = 0$$

$$\text{or } \bar{a}\bar{z}^2 + \bar{b}\bar{z} + \bar{c} = 0$$

$$\text{or } \bar{a}z^2 - \bar{b}z + \bar{c} = 0 \quad (4)$$

Equations (3) and (4) must be identical as their roots are same.

$$\therefore \frac{a}{\bar{a}} = -\frac{b}{\bar{b}} = \frac{c}{\bar{c}}$$

$$\Rightarrow a\bar{c} = \bar{a}c, \bar{a}b + \bar{a}b = 0 \text{ and } b\bar{c} + \bar{b}c = 0$$

Hence, $a\bar{c}$ is purely real and $\bar{a}b$ and $b\bar{c}$ are purely imaginary.

Let z_1 (purely real) be a root of the given equation. Then,

$$z_1 = \bar{z}_1 \text{ and } az_1^2 + bz_1 + c = 0 \quad (5)$$

$$\text{or } \overline{az_1^2 + bz_1 + c} = 0$$

$$\text{or } \bar{a}\bar{z}_1^2 + \bar{b}\bar{z}_1 + \bar{c} = 0$$

$$\text{or } \bar{a}z_1^2 + \bar{b}z_1 + \bar{c} = 0 \quad (6)$$

Now (5) and (6) must have one common root. Hence,

$$(c\bar{a} - a\bar{c})^2 = (b\bar{c} - c\bar{b})(\bar{a}b - a\bar{b})$$

17. (4) We know that

$$|z + i\omega| \leq |z| + |i\omega| = |z| + |\omega| \leq 2$$

$$\text{Given } |z + i\omega| = 2 \Rightarrow |z| = |\omega| = 1.$$

18. (4) Suppose $z = x + iy$, $\omega = \alpha + i\beta$

$$|z + i\omega| = 2$$

$$\Rightarrow |z|^2 + |\omega|^2 + i\omega\bar{z} - i\bar{\omega}z = 4$$

$$\Rightarrow i\omega\bar{z} - i\bar{\omega}z = 2 \quad (1)$$

$$\text{Also } |z - i\bar{\omega}| = 2$$

$$\Rightarrow |z|^2 + |\omega|^2 + i\omega z - i\bar{\omega}\bar{z} = 4$$

$$\Rightarrow i\omega z - i\bar{\omega}\bar{z} = 2 \quad (2)$$

Adding (1) and (2), we get

$$\Rightarrow i(\omega - \bar{\omega})(z + \bar{z}) = 4$$

$$\Rightarrow i(2i\beta)(2x) = 4$$

$$\Rightarrow \beta x = -1 \quad (3)$$

Subtracting (1) from (2), we get

$$i(\omega + \bar{\omega})(z - \bar{z}) = 0 \Rightarrow \alpha y = 0 \quad (4)$$

$$\Rightarrow \text{either } \alpha = 0 \text{ or } y = 0$$

If $y = 0$, then from $x^2 + y^2 = 1$,

$$x = \pm 1 \text{ and } z = 1 \text{ or } -1$$

If $\alpha = 0$ then from $\alpha^2 + \beta^2 = 1$,

$$\beta = \pm 1 \Rightarrow \omega = \pm i$$

So $\text{Im}(z) = \text{Re}(\omega) = 0$.

19. (3) As obtained earlier $\omega = \pm i$.

20. (3) 21. (4), 22. (2)

Given equation of line is $a\bar{z} + \bar{a}z + b = 0 \forall b \in \mathbb{R}$.

Let the PQ be the segment intercepted between axes.

For intercept on real axis Z_R ,

$$z = \bar{z}$$

$$\Rightarrow Z_R(a + \bar{a}) + b = 0$$

$$\Rightarrow Z_R = \frac{-b}{a + \bar{a}}$$

For intercept on imaginary Z_I ,

$$z + \bar{z} = 0$$

$$\Rightarrow Z_I(\bar{a} - a) + b = 0$$

$$\Rightarrow Z_I = -\frac{b}{\bar{a} - a}$$

For mid-point,

$$z = \frac{Z_R + Z_I}{2}$$

$$\Rightarrow z = \frac{-b}{2} \left[\frac{1}{\bar{a} + a} + \frac{1}{\bar{a} - a} \right]$$

$$\Rightarrow z = \frac{\bar{a}b}{(a + \bar{a})(a - \bar{a})}$$

$$\Rightarrow z = \frac{\bar{a}b}{a^2 - (\bar{a})^2}$$

$$\Rightarrow \frac{z[a^2 - (\bar{a})^2]}{\bar{a}} = b \text{ (= real)}$$

$$\Rightarrow z \frac{a^2 - (\bar{a})^2}{\bar{a}} = \bar{z} \frac{(\bar{a})^2 - (a)^2}{a}$$

$$\Rightarrow az + \bar{a}\bar{z} = 0$$

23. (3), 24. (2), 25. (4)

$$az + b\bar{z} + c = 0 \quad (1)$$

$$\text{or } \bar{a}\bar{z} + \bar{b}z + \bar{c} = 0 \quad (2)$$

Eliminating $\bar{z}$ from (1) and (2), we get

$$z = \frac{c\bar{a} - b\bar{c}}{|b|^2 - |a|^2}$$

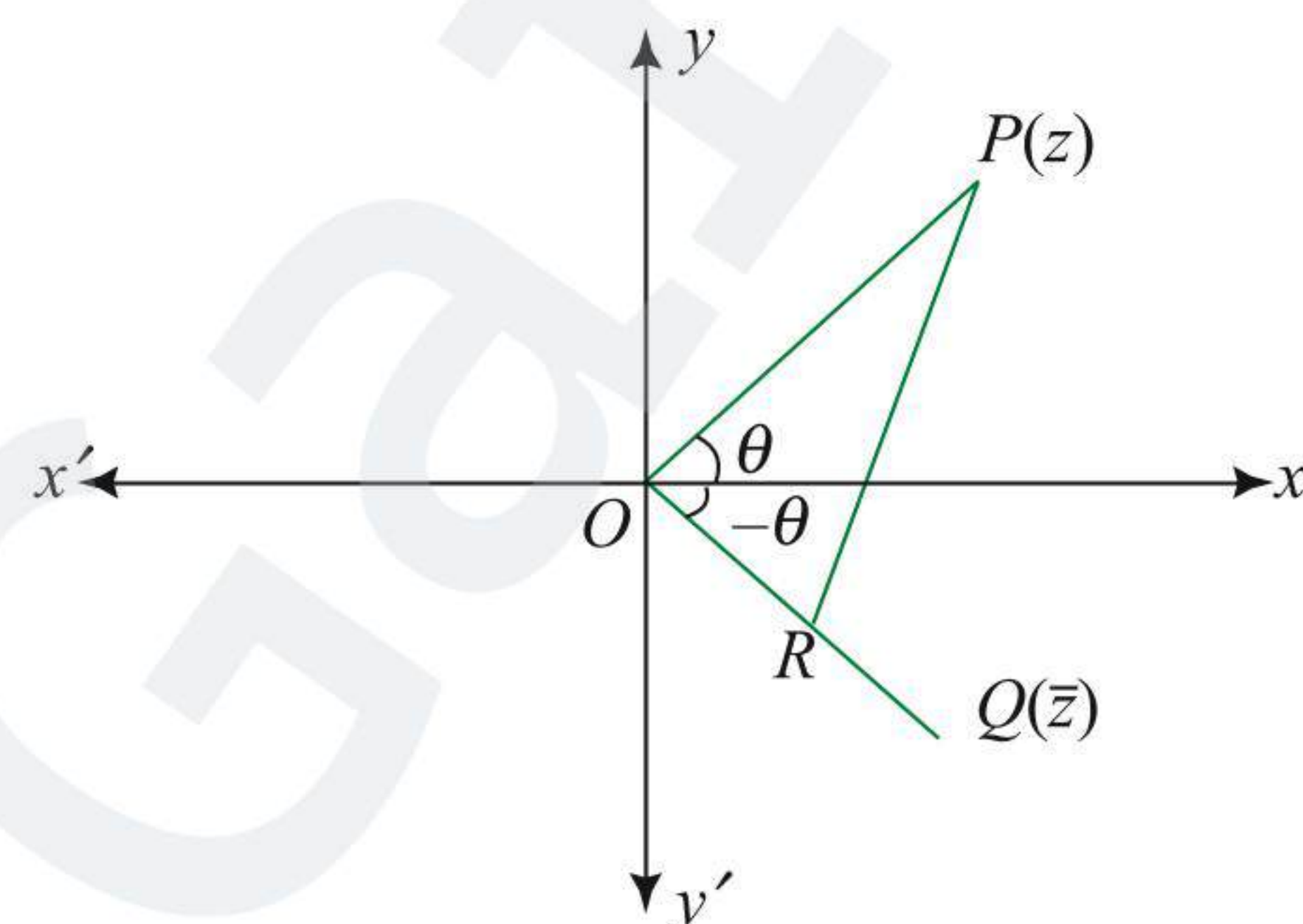
If $|a| \neq |b|$, then z represents one point on the Argand plane. If $|a| = |b|$ and $\bar{a}c \neq b\bar{c}$, then no such z exists. Adding (1) and (2),

$$(\bar{a} + b)\bar{z} + (a + \bar{b})z + (c + \bar{c}) = 0$$

This is of the form $A\bar{z} + \bar{A}z + B = 0$, where $B = c + \bar{c}$ is real.

Hence, locus of z is a straight line.

26. (1), 27. (2), 28. (2)



$$\text{We have } \left| |z| - \left| \frac{4}{z} \right| \right| \leq \left| z - \frac{4}{z} \right| = 2$$

$$\Rightarrow -2 \leq |z| - \frac{4}{|z|} \leq 2$$

$$\Rightarrow |z|^2 + 2|z| - 4 \geq 0 \text{ and } |z|^2 - 2|z| - 4 \leq 0$$

$$\Rightarrow (|z| + 1)^2 - 5 \geq 0 \text{ and } (|z| - 1)^2 \leq 5$$

$$\Rightarrow (|z| + 1 + \sqrt{5})(|z| + 1 - \sqrt{5}) \geq 0$$

$$\Rightarrow |z| \geq \sqrt{5} - 1$$

$$\text{and } (|z| - 1 + \sqrt{5})(|z| - 1 - \sqrt{5}) \leq 0$$

$$\Rightarrow \sqrt{5} - 1 \leq |z| \leq \sqrt{5} + 1$$

$$\Rightarrow \sqrt{5} - 1 \leq |z| \leq \sqrt{5} + 1$$

Hence, the least modulus is $\sqrt{5} - 1$ and the greatest modulus is $\sqrt{5} + 1$. Also,

$$|z| = \sqrt{5} + 1$$

$$\Rightarrow \frac{4}{|z|} = \sqrt{5} - 1$$

$$\text{Now, } \frac{4}{z} = \frac{4\bar{z}}{|z|^2}$$

Hence, $4/z$ lies in the direction of $\bar{z}$.

$$\left| z - \frac{4}{z} \right| = PR = 2 \text{ (given)}$$

We have

$$OP = \sqrt{5} + 1 \text{ and } OR = \sqrt{5} - 1$$

$$\begin{aligned} \Rightarrow \cos 2\theta &= \frac{OP^2 + OR^2 - PR^2}{2 \cdot OP \cdot OR} \\ &= \frac{(\sqrt{5} + 1)^2 + (\sqrt{5} - 1)^2 - 4}{2(5 - 1)} = 1 \end{aligned}$$

$$\Rightarrow 2\theta = 0, 2\pi$$

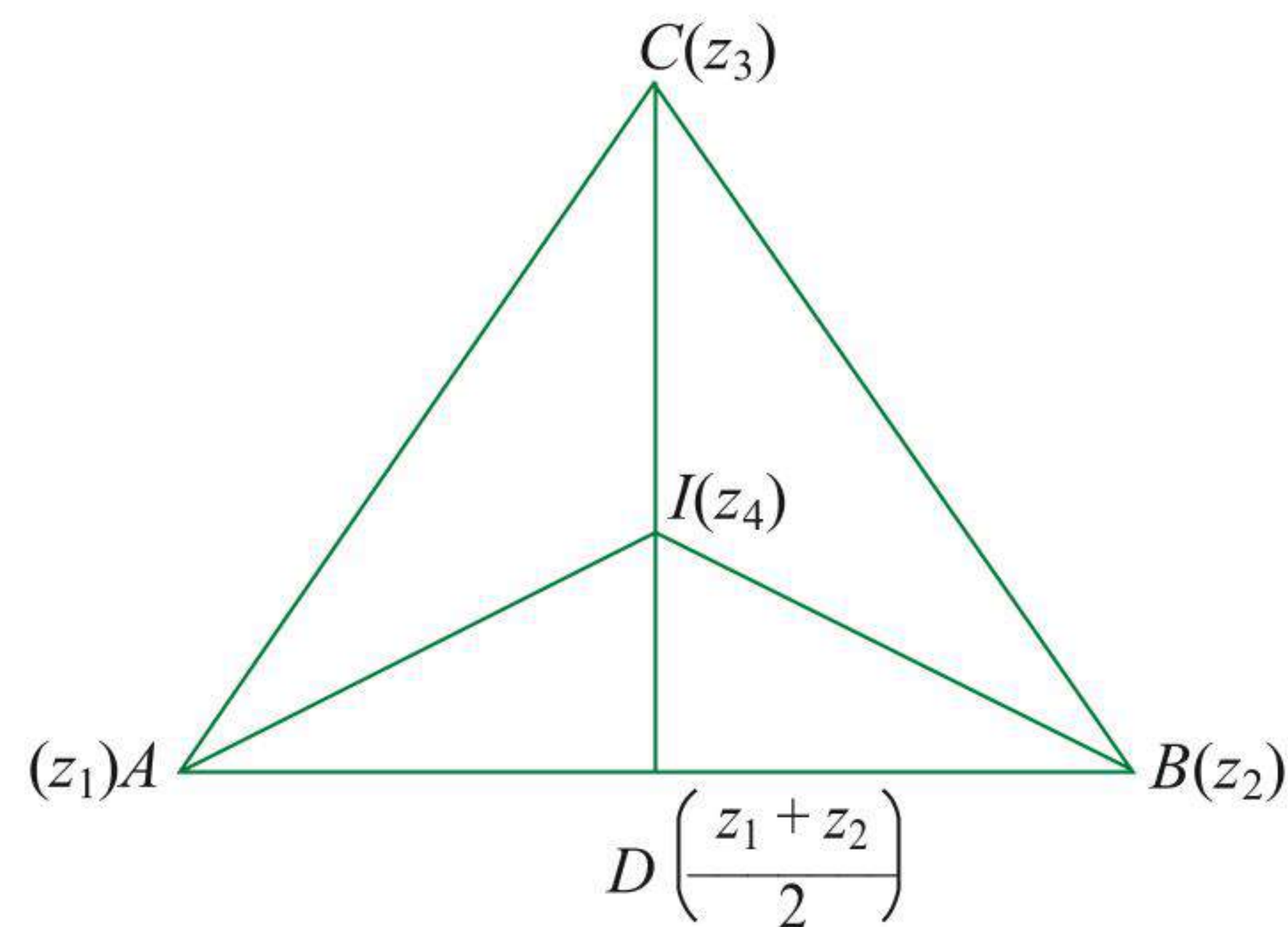
$$\Rightarrow \theta = 0, \pi$$

$$\Rightarrow z \text{ is purely real}$$

$$\Rightarrow z = \pm(\sqrt{5} + 1)$$

Similarly for $|z| = \sqrt{5} - 1$, we have $z = \pm(\sqrt{5} - 1)$.

29. (1)



$$\frac{AB \times AC}{(IA)^2} = \frac{AB}{IA} \times \frac{AC}{IA}$$

$$\angle IAB = \frac{\theta}{2}, \angle IAC = \frac{\theta}{2}$$

$$\frac{z_2 - z_1}{z_4 - z_1} = \frac{|z_2 - z_1|}{|z_4 - z_1|} e^{-i\frac{\theta}{2}}$$

and $\frac{z_3 - z_1}{z_4 - z_1} = \frac{|z_3 - z_1|}{|z_4 - z_1|} e^{i\frac{\theta}{2}}$

Multiplying,

$$\frac{z_2 - z_1}{z_4 - z_1} \frac{z_3 - z_1}{z_4 - z_1} = \frac{|z_2 - z_1|}{|z_4 - z_1|} \frac{|z_3 - z_1|}{|z_4 - z_1|}$$

$$\Rightarrow \frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2} = \frac{AB \times AC}{IA^2} \quad (1)$$

30. (2) From (1),

$$\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2} = 2 \left(\frac{AD}{IA} \right)^2 \left(\frac{AC}{AD} \right) \quad (\because AB = 2 AD)$$

$$\begin{aligned} \Rightarrow (z_2 - z_1)(z_3 - z_1) &= (z_4 - z_1)^2 2 \cos^2 \frac{\theta}{2} \sec \theta \\ &= (z_4 - z_1)^2 (\cos \theta + 1) \sec \theta \end{aligned}$$

31. (3) Keeping in mind that $\tan \theta = CD/AD$ and $\tan \theta/2 = ID/BD$, we have,

$$\frac{z_3 - \frac{z_1 + z_2}{2}}{z_1 - \frac{z_1 + z_2}{2}} = \frac{\left| z_3 - \frac{z_1 + z_2}{2} \right|}{\left| z_1 - \frac{z_1 + z_2}{2} \right|} e^{-i\frac{\pi}{2}}$$

$$\Rightarrow \frac{2z_3 - z_1 - z_2}{z_1 - z_2} = \frac{CD}{AD} e^{-i\frac{\pi}{2}} \quad (1)$$

and $\frac{z_4 - \frac{z_1 + z_2}{2}}{z_2 - \frac{z_1 + z_2}{2}} = \frac{\left| z_4 - \frac{z_1 + z_2}{2} \right|}{\left| z_2 - \frac{z_1 + z_2}{2} \right|} e^{i\frac{\pi}{2}}$

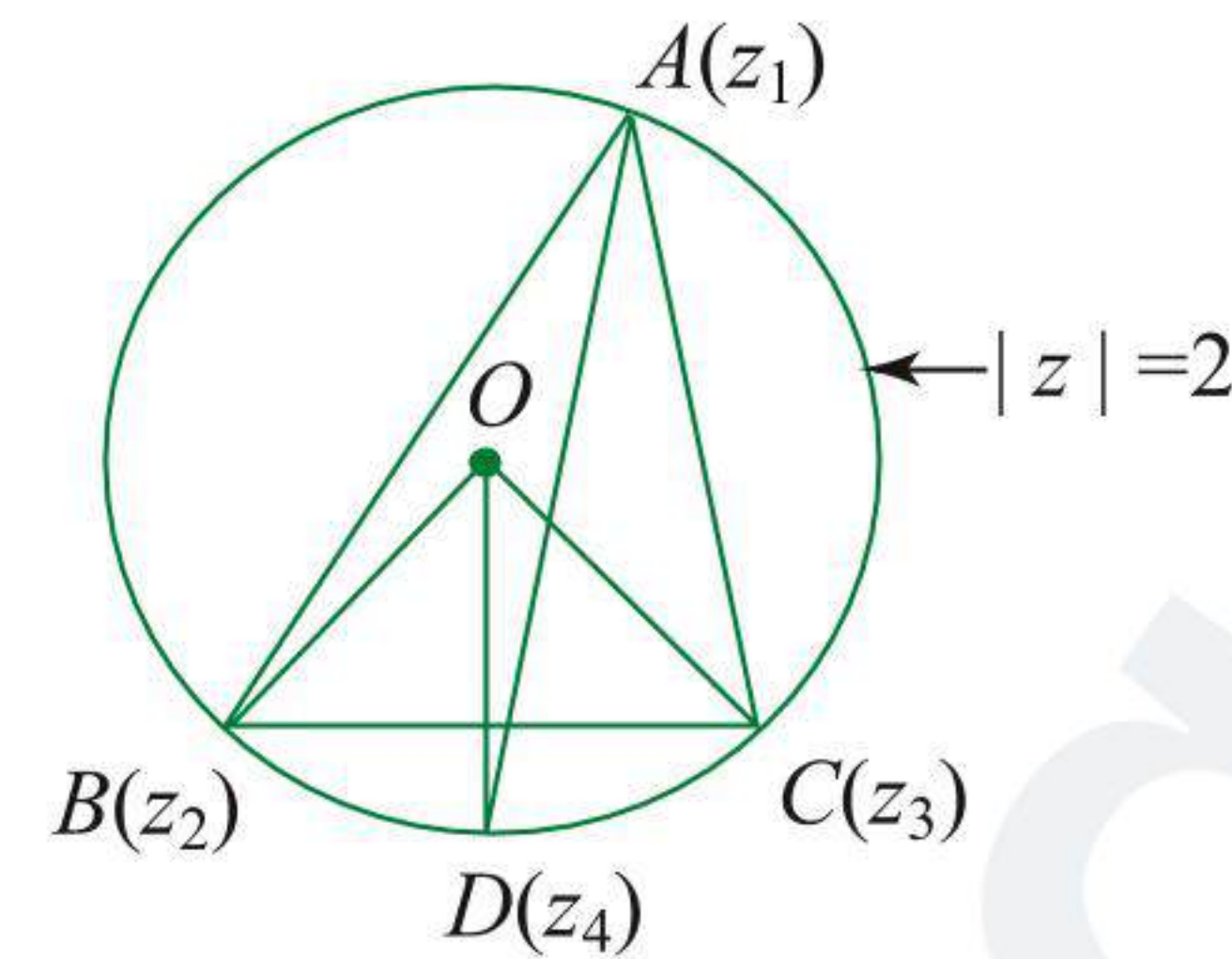
$$\Rightarrow \frac{2z_4 - z_1 - z_2}{z_2 - z_1} = \frac{ID}{BD} e^{i\frac{\pi}{2}} \quad (2)$$

Multiplying (1) and (2), we have

$$\frac{2z_3 - z_1 - z_2}{z_1 - z_2} \frac{2z_4 - z_1 - z_2}{z_2 - z_1} = \frac{CD}{AD} \frac{ID}{BD} = \tan \theta \tan \frac{\theta}{2}$$

$$\Rightarrow (z_2 - z_1)^2 \tan \theta \tan \frac{\theta}{2} = -(z_1 + z_2 - 2z_3)(z_1 + z_2 - 2z_4)$$

32. (4) $\angle BOD = 2\angle BAD = A$
 $\angle COD = 2\angle CAD = A$



$$\frac{z_4}{z_2} = e^{iA}, \frac{z_3}{z_4} = e^{iA} \quad (\text{From rotation about the point } O)$$

$$\Rightarrow z_4^2 = z_2 z_3$$

33. (3) Clearly, OD bisects $\angle BOC$ of isosceles triangle BOC . Thus angle between segments OD and BC is $\pi/2$.

$$\therefore \arg \left(\frac{z_4}{z_2 - z_3} \right) = \frac{\pi}{2}$$

34. (3)

Matrix Match Type

1. $a \rightarrow q$; $b \rightarrow s$; $c \rightarrow p$; $d \rightarrow r$

Since graph is concave upward, $a > 0$.

Graph intersects y -axis at negative y -axis. So, $c < 0$.

Also, vertex lies to the right of y -axis.

$$\text{So, } -\frac{b}{2a} > 0 \quad \therefore b < 0$$

Complex number $a + ib$ lies in fourth quadrant.

$$\therefore \arg(a + ib) = \tan^{-1} \frac{b}{a}$$

Complex number $ac - ibc$ lies in third quadrant.

$$\therefore \arg(ac - ibc) = -\pi - \tan^{-1} \frac{b}{a}$$

Complex number $-ac + ibc$ lies in first quadrant.

$$\therefore \arg(-ac + ibc) = -\tan^{-1} \frac{b}{a}$$

Complex number $abc - ib^2c$ lies in second quadrant.

$$\therefore \arg(abc - ib^2c) = \pi + \tan^{-1} \frac{b}{a}$$

2. $a \rightarrow s$; $b \rightarrow p$; $c \rightarrow q$; $d \rightarrow r$

a. $\frac{z_3 - z_1}{z_2 - z_1} = \frac{1 + i\sqrt{3}}{2}$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = e^{i\frac{\pi}{3}}$$

$$\Rightarrow \left| \frac{z_3 - z_1}{z_2 - z_1} \right| = 1 \text{ and } \arg \left(\frac{z_3 - z_1}{z_2 - z_1} \right) = \frac{\pi}{3}$$

So, triangle is equilateral.

b. $\operatorname{Re} \left(\frac{z_3 - z_1}{z_3 - z_2} \right) = 0$

$$\Rightarrow \frac{z_3 - z_1}{z_3 - z_2} \text{ is purely imaginary}$$

$$\Rightarrow \arg\left(\frac{z_3 - z_1}{z_3 - z_2}\right) = \pm \frac{\pi}{2}$$

So, triangle is right angled.

c. Let $\frac{z_3 - z_1}{z_3 - z_2} = r(\cos\theta + i\sin\theta)$

Since $\cos\theta < 0$, $\arg\left(\frac{z_3 - z_1}{z_3 - z_2}\right) > \frac{\pi}{2}$

So, triangle is obtuse angled.

d. $\frac{z_3 - z_1}{z_3 - z_2} = i$

$$\Rightarrow |z_3 - z_1| = |z_3 - z_2|$$

and $\arg\left(\frac{z_3 - z_1}{z_3 - z_2}\right) = \frac{\pi}{2}$

So, triangle is isosceles right angled.

3. a $\rightarrow$ q; b $\rightarrow$ p; c $\rightarrow$ s; d $\rightarrow$ r

a. Minimum value is equal to distance between $(0, 0)$ and $(\cos\alpha, \sin\alpha)$ which is equal to 1.

b. $|z - 1| + |z + 1| = 2$ represents the line segment joining points '1' and '-1'. $|z + 2| + |z - 2| = 8$ represents the ellipse with foci '-2' and '2'. Clearly, line segment does not intersect the ellipse.

c. $(z_1 - 3z_2)(\bar{z}_1 - 3\bar{z}_2) = (3 - z_1\bar{z}_2)(3 - \bar{z}_1z_2)$

$$\Rightarrow |z_1|^2 - |z_1|^2|z_2|^2 + 9|z_2|^2 - 9 = 0$$

$$\Rightarrow (1 - |z_2|^2)(|z_1|^2 - 9) = 0$$

$$\Rightarrow |z_1| = 3$$

d. $z = \frac{(2\sqrt{3} + 2i)^8}{(1 - i)^6} + \frac{(1 + i)^6}{(2\sqrt{3} - 2i)^8}$

$$= \frac{z_1^8}{\bar{z}_2^6} + \frac{z_2^6}{\bar{z}_1^8} = \frac{|z_1|^{16} + |z_2|^{12}}{(\bar{z}_2)^6(\bar{z}_1)^8}$$

$$\begin{aligned} \Rightarrow \arg(z) &= -8\arg(\bar{z}_1) - 6\arg(\bar{z}_2) \\ &= 8\arg(z_1) + 6\arg(z_2) \\ &= 8 \cdot \frac{\pi}{6} + 6 \cdot \frac{\pi}{4} = \frac{4\pi}{3} + \frac{3\pi}{2} = \frac{17\pi}{6} \end{aligned}$$

$$\therefore |4\sin\theta| = 4\left|\sin\frac{17\pi}{6}\right| = 2$$

4. a $\rightarrow$ r; b $\rightarrow$ p; c $\rightarrow$ s; d $\rightarrow$ q.

We have equation $|z - 5| + m|z - 12i| = n$

a. $m = 1$ and $n = 13$.

$$\therefore |z - 5| + |z - 12i| = 13$$

$$\text{or } |z - 5| + |z - 12i| = |5 - 12i|$$

So, z lies on the line segment joining complex numbers '5' and '12i'.

b. $m = 1$, $n = 15$

$$\therefore |z - 5| + |z - 12i| = 15 > 13$$

So, z lies on the ellipse whose foci are '5' and '12i'.

c. $m = -1$, $n = 13$

$$\therefore ||z - 5| - |z - 12i|| = 13$$

$$\text{or } ||z - 5| - |z - 12i|| = |5 - 12i|$$

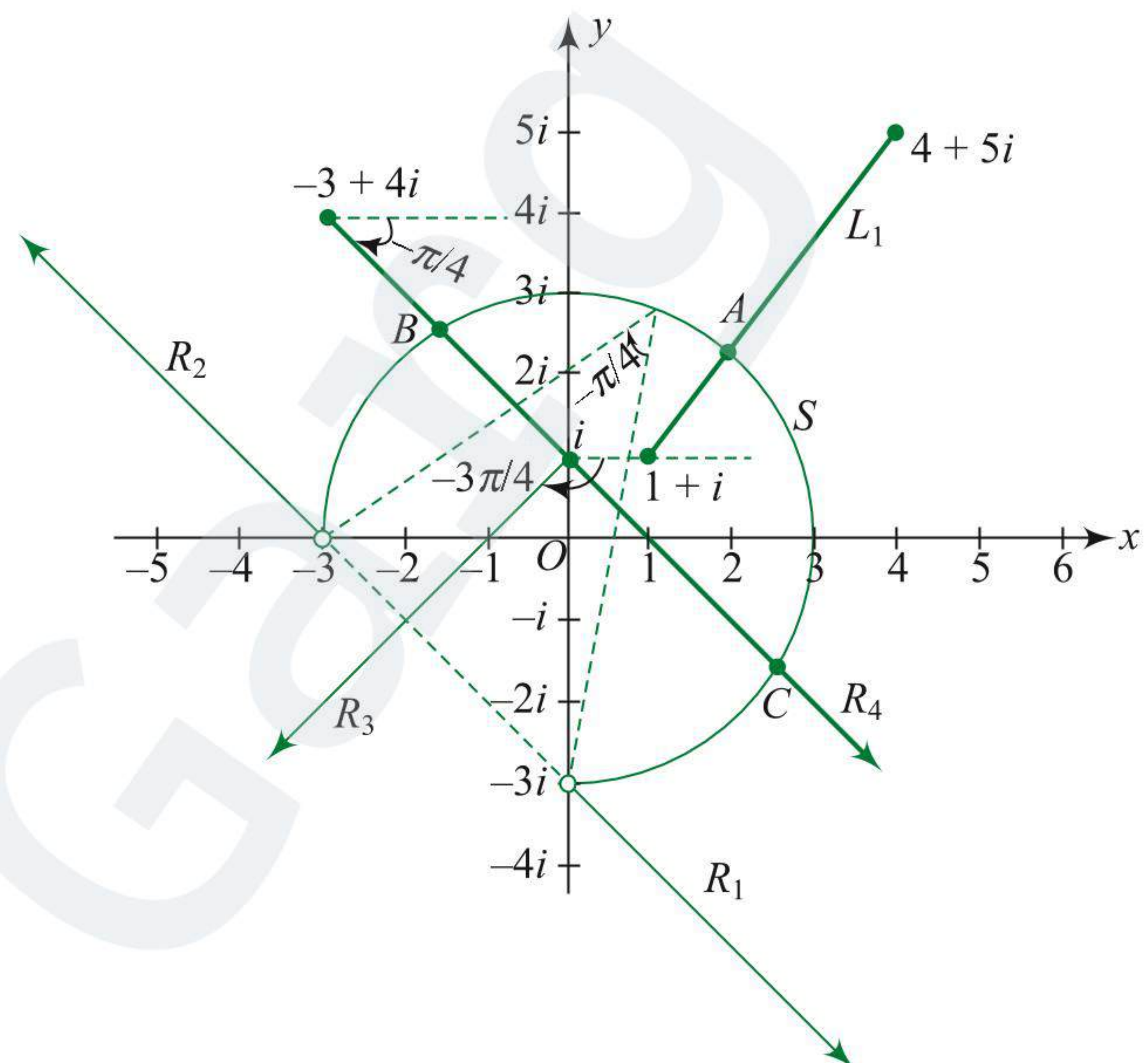
So, z lies on either of the two rays emanating from '5' and '12i' and moving in opposite directions.

d. $m = -1$, $n = 12$

$$\therefore ||z - 5| - |z - 12i|| = 12 < 13$$

So, z lies on the hyperbola whose foci are '5' and '12i'.

5. (1) $\arg\frac{z+3}{z+3i} = -\frac{\pi}{4}$ is the major arc excluding complex numbers '-3' and '-3i'.



a. $||z + 3| - |z + 3i|| = 3\sqrt{2} = |3 - 3i|$
This implies that z lies either on ray R_1 emanating from '-3i' or ray R_2 emanating from '-3'.

So, there is no point of intersection.

b. $|z - (1 + i)| + |z - (4 + 5i)| = 5 = |(4 + 5i) - (1 + i)|$
This implies that z lies on the line segment L_1 joining complex numbers '1 + i' and '4 + 5i'.
Clearly, there is one point of intersection.

c. $\arg(z - i) = -\frac{3\pi}{4}$

So, z lies on the ray R_3 emanating from 'i' and making an angle $-3\pi/4$ with horizontal. Clearly, there is no point of intersection.

d. $\arg(z - (-3 + 4i)) = -\frac{\pi}{4}$

So, z lies on the ray R_4 emanating from $-3 + 4i$ and making an angle $-\pi/4$ with horizontal. Clearly, there are two points of intersection.

6. (2) We have $z^{27} - 1 = 0$.

The roots of the equation are $\cos\frac{2k\pi}{27} + i\sin\frac{2k\pi}{27}$; $k = 0, 1, 2, 3, \dots, 26$.

This equation has one real root '1'. The other 26 roots are in the form of 13 conjugate pairs.

Of these, 13 roots lie above real-axis and the other 13 are lie below real axis.

So, there are 13 roots having negative imaginary part.

The other equation is $z^{36} - 1 = 0$.

This equation has two real roots, ± 1 and two purely imaginary roots, $\pm i$.

Of remaining 32 roots, 16 lie to the right of imaginary axis.

So, there are 17 roots having positive real part.

The roots of the equation are $\cos \frac{2k\pi}{36} + i \sin \frac{2k\pi}{36}$; $k = 0, 1, 2, 3, \dots, 35$.

Number of common roots = HCF(27, 36) = 9

$$\therefore n(A \cap B) = 9$$

$$\therefore n(A \cup B) = 27 + 36 - 9 = 54$$

Numerical Value Type

$$1. (3) \quad x = \frac{x^3}{x^2} = \frac{2+11i}{3+4i} \times \frac{3-4i}{3-4i} = \frac{50+25i}{25} = 2+i$$

2. (21) We have

$$x^3 - y^3 = 98i$$

$$\text{or } (x-y)^3 + 3xy(x-y) = 98i$$

$$\text{or } -343i + 3(a+ib)(7i) = 98i$$

$$\text{or } -343 + 3(a+bi)7 = 98$$

$$\text{or } a+ib = 21$$

$$\Rightarrow a = 21 \text{ and } b = 0$$

$$\Rightarrow a + b = 21$$

3. (1) We have $x = \omega - \omega^2 - 2$ or $x + 2 = \omega - \omega^2$

$$\text{Squaring, } x^2 + 4x + 4 = \omega^2 + \omega^4 - 2\omega^3 = \omega^2 + \omega - 2 = -3$$

$$\Rightarrow x^2 + 4x + 7 = 0.$$

Dividing $x^4 + 3x^3 + 2x^2 - 11x - 6$ by $x^2 + 4x + 7$, we get

$$\begin{aligned} x^4 + 3x^3 + 2x^2 - 11x - 6 &= (x^2 + 4x + 7)(x^2 - x - 1) + 1 \\ &= (0)(x^2 - x - 1) + 1 = 0 + 1 = 1 \end{aligned}$$

$$4. (15) \quad z^2 = 81 - b^2 + 18bi$$

$$z^3 = 729 + 243bi - 27b^2 - b^3i$$

$$\Rightarrow 243b - b^3 = 18b$$

$$\Rightarrow b = 15$$

5. (2) $\bar{z} + z = 0$

$$\Rightarrow \bar{z} = -z$$

$$\text{Now } |z|^2 - 4zi = z^2$$

$$\Rightarrow -z^2 - 4zi = z^2$$

$$\Rightarrow 2z = -4i$$

$$\Rightarrow z = -2i$$

$$\Rightarrow |z| = 2$$

6. (3) $(1+ri)^3 = s(1+i)$

$$\Rightarrow 1 + 3ri + 3r^2i^2 + r^3i^3 = s(1+i)$$

$$\Rightarrow 1 - 3r^2 + i(3r - r^3) = s + si$$

$$\Rightarrow 1 - 3r^2 = s = 3r - r^3$$

$$\text{Hence, } 1 - 3r^2 = 3r - r^3$$

$$\Rightarrow r^3 - 3r^2 - 3r + 1 = 0$$

Thus, sum of three roots is 3.

7. (4) We have $|z|^2 + \frac{16}{|z|^3} = z^2 - 4z = \bar{z}^2 - 4\bar{z}$

$$\Rightarrow (z - \bar{z})(z + \bar{z} - 4) = 0$$

$$\Rightarrow z = \bar{z} = x \quad (x \neq 2)$$

$$\text{So, } x^2 = 4x + x^2 + \frac{16}{|x|^3}$$

$$\Rightarrow x = \frac{-4}{|x|^3}$$

$$\Rightarrow x = -\sqrt{2}$$

$$\therefore z = -\sqrt{2}$$

$$\therefore |z|^4 = 4$$

8. (17) Let $z = a + bi$.

$$\Rightarrow |z|^2 = a^2 + b^2.$$

$$\text{Now } z + |z| = 2 + 8i$$

$$\Rightarrow a + bi + \sqrt{a^2 + b^2} = 2 + 8i$$

$$\Rightarrow a + \sqrt{a^2 + b^2} = 2, b = 8$$

$$\Rightarrow a + \sqrt{a^2 + 64} = 2$$

$$\Rightarrow a^2 + 64 = (2-a)^2 = a^2 - 4a + 4,$$

$$\Rightarrow 4a = -60, a = -15.$$

$$\text{Thus, } a^2 + b^2 = 225 + 64 = 289$$

$$\therefore |z| = \sqrt{a^2 + b^2} = \sqrt{289} = 17$$

9. (1) Let $z = a + ib$

$$\text{Given } |z| = 2$$

$$\Rightarrow a^2 + b^2 = 4 \Rightarrow a, b \in [-2, 2]$$

$$\text{Now } w = \frac{(a+1)+ib}{(a-1)+ib};$$

$$\Rightarrow |w| = \sqrt{\frac{(a+1)^2 + b^2}{(a-1)^2 + b^2}} = \sqrt{\frac{a^2 + b^2 + 2a + 1}{a^2 + b^2 - 2a + 1}} = \sqrt{\frac{5+2a}{5-2a}}$$

$$|w|_{\max} = \sqrt{\frac{5+4}{1}} = 3 \quad (\text{when } a = 2)$$

$$|w|_{\min} = \sqrt{\frac{5-4}{9}} = \frac{1}{3} \quad (\text{when } a = -2)$$

Hence, required product is 1.

10. (1) $z^4 + z^3 + z^2 + z + 1 = 0$

$$\text{or } (z^2(z^2 + z + 1) + (z^2 + z + 1)) = 0$$

$$\text{or } (z^2 + z + 1)(z^2 + 1) = 0$$

$$\therefore z = i, -i, \omega, \omega^2. \text{ For each, } |z| = 1$$

11. (5) Roots are $2\omega, (2+3\omega), (2+3\omega^2), (2-\omega-\omega^2)$

$2+3\omega$ and $2+3\omega^2$ are conjugate to each other.

2ω is complex root, then other root must be $2\omega^2$ (as complex roots occur in conjugate pair)

$$2 - \omega - \omega^2 = 2 - (-1) = 3 \text{ which is real.}$$

Hence, least degree of the polynomial is 5.

12. (6) We have $|a\omega + b| = 1$

$$\Rightarrow |a\omega + b|^2 = 1$$

$$\Rightarrow (a\omega + b)(a\bar{\omega} + \bar{b}) = 1$$

$$\Rightarrow a^2 + ab(\omega + \bar{\omega}) + b^2 = 1$$

$$\Rightarrow a^2 - ab + b^2 = 1$$

$$\Rightarrow (a-b)^2 + ab = 1 \quad (1)$$

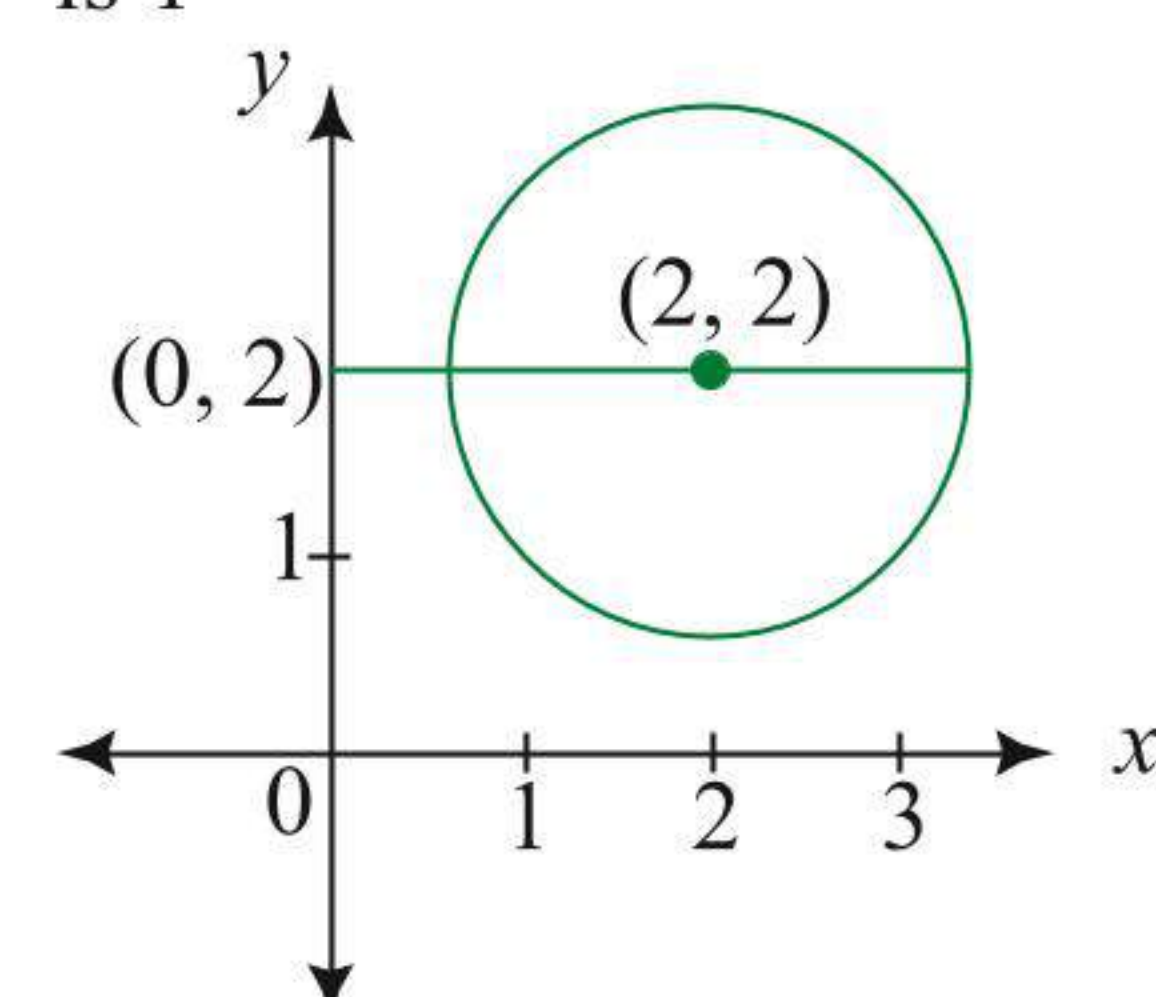
when $(a-b)^2 = 0$ and $ab = 1$, then $(1, 1); (-1, -1)$

when $(a-b)^2 = 1$ and $ab = 0$, then $(0, 1); (1, 0); (0, -1); (-1, 0)$

Hence, there are 6 ordered pairs.

13. (3) $|z - 2 - 2i| \leq 1$

denotes the region inside a circle with center $(2, 2)$ and radius is 1



$$\begin{aligned}
 |2iz + 4| &= |2i(z - 2i)| \\
 &= |2i| |z - 2i| \\
 &= 2|z - 2i|
 \end{aligned}$$

$|z - 2i|$ = distance of z from $P(0, 2)$

Hence, maximum value is 3.

$$\begin{aligned}
 \text{14. (20)} \quad |3z + 9 - 7i| &= |(3z + 6 - 3i) + (3 - 4i)| \leq |3z + 6 - 3i| + |3 - 4i| \\
 &= 3|z + 2 - i| + 5 \\
 &= 20
 \end{aligned}$$

$$\begin{aligned}
 \text{15. (125)} \quad Z_1 &= (8 \sin \theta + 7 \cos \theta) + i(\sin \theta + 4 \cos \theta) \\
 Z_2 &= (\sin \theta + 4 \cos \theta) + i(8 \sin \theta + 4 \cos \theta) \\
 \text{Hence, } Z_1 &= x + iy \text{ and } Z_2 = y + ix \\
 \text{where } x &= (8 \sin \theta + 7 \cos \theta) \text{ and } y = (\sin \theta + 4 \cos \theta) \\
 Z_1 \cdot Z_2 &= (xy - xy) + i(x^2 + y^2) = i(x^2 + y^2) = a + ib \\
 \Rightarrow a &= 0; b = x^2 + y^2 \\
 \text{Now, } x^2 + y^2 &= (8 \sin \theta + 7 \cos \theta)^2 + (\sin \theta + 4 \cos \theta)^2 \\
 &= 65 \sin^2 \theta + 65 \cos^2 \theta + 120 \sin \theta \cdot \cos \theta \\
 &= 65 + 60 \sin 2\theta \\
 \Rightarrow Z_1 \cdot Z_2|_{\max} &= 125
 \end{aligned}$$

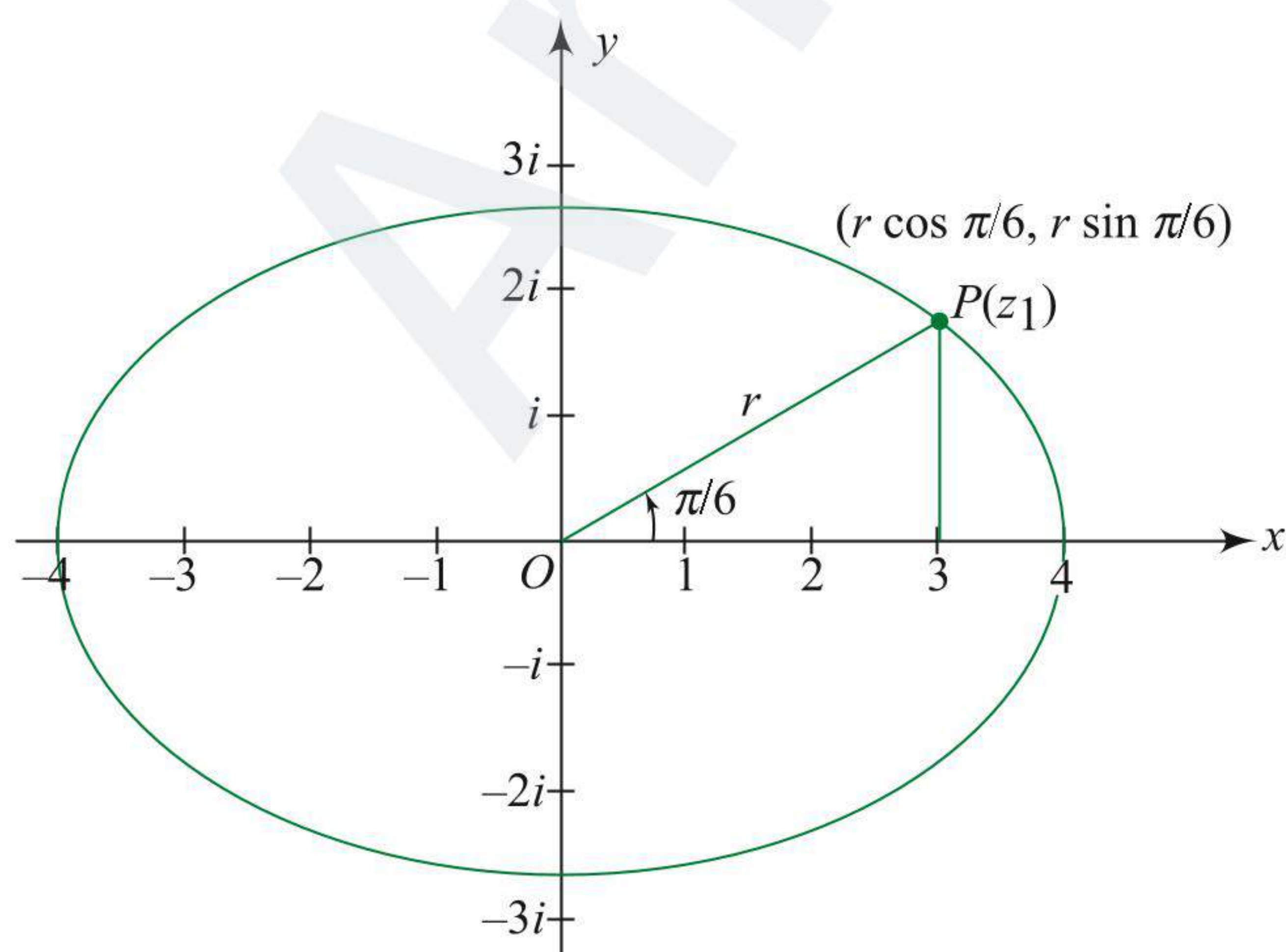
$$\begin{aligned}
 \text{16. (18)} \quad A &\equiv (1 + 2i)x^3 - 2(3 + i)x^2 + (5 - 4i)x + 2a^2 = 0 \\
 \text{Let the real root of equation be } \alpha. &\text{ Then} \\
 (1 + 2i)\alpha^3 - 2(3 + i)\alpha^2 + (5 - 4i)\alpha + 2a^2 &= 0 \\
 \text{Equating imaginary part zero, we get} \\
 2\alpha^3 - 2\alpha^2 - 4\alpha &= 0 \\
 \Rightarrow \text{or } \alpha(\alpha^2 - \alpha - 2) &= 0 \\
 \Rightarrow \alpha = 0 \text{ or } \alpha = -1, 2
 \end{aligned}$$

Now equating real part zero, we have

$$\begin{aligned}
 \alpha^3 - 6\alpha^2 + 5\alpha + 2a^2 &= 0 \\
 \alpha = 0 &\Rightarrow a = 0 \\
 \alpha = -1 &\Rightarrow a = \pm \sqrt{6} \\
 \alpha = 2 &\Rightarrow a = \pm \sqrt{3} \\
 \Rightarrow \sum a^2 &= (0)^2 + (\sqrt{6})^2 + (-\sqrt{6})^2 + (\sqrt{3})^2 + (-\sqrt{3})^2 \\
 &= 18
 \end{aligned}$$

$$\begin{aligned}
 \text{17. (30)} \quad \text{Let } z &= x + iy \\
 \therefore E &= z\bar{z} + (z - 3)(\bar{z} - 3) + (z - 6i)(\bar{z} + 6i) \\
 &= 3z\bar{z} - 3(z + \bar{z}) + 9 + 6(z - \bar{z})i + 36 \\
 &= 3(x^2 + y^2) - 6x - 12y + 45 \\
 &= 3[x^2 + y^2 - 2x - 4y + 15] \\
 &= 3[(x - 1)^2 + (y - 2)^2 + 10] \\
 \therefore E_{\min} &= 30 \text{ when } x = 1 \text{ and } y = 2
 \end{aligned}$$

$$\text{18. (448)} \quad |z - 3| + |z + 3| = 8$$



Given locus is ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with length of major axis

$$2a = 8 \text{ and } e = \frac{6}{8} = \frac{3}{4}.$$

$$\therefore b^2 = a^2(1 - e^2) = 7$$

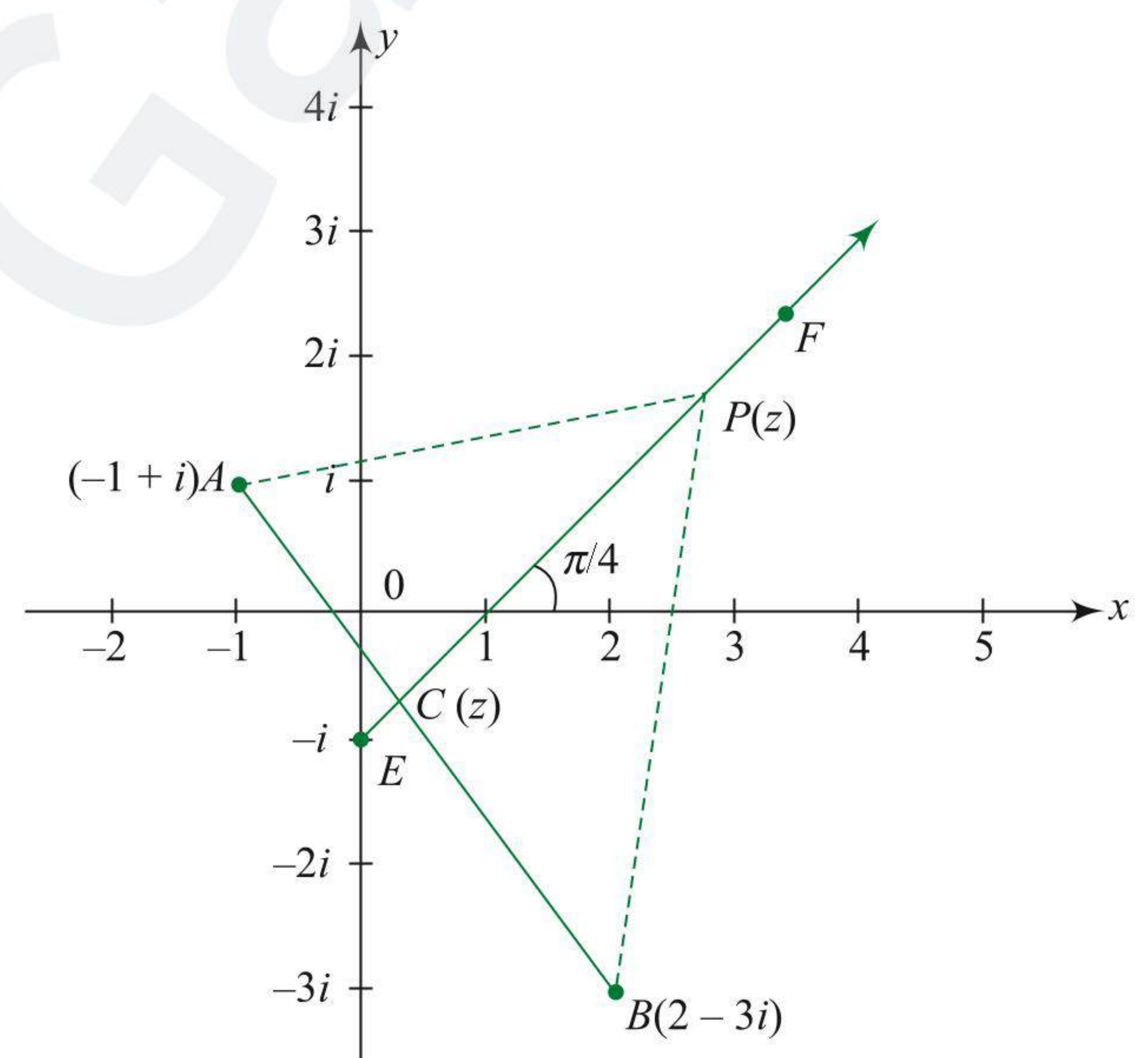
Therefore, ellipse is $\frac{x^2}{16} + \frac{y^2}{7} = 1$.

Let $|z_1| = r$. Then point $P(z_1)$

$$\Rightarrow \left(r \cos \frac{\pi}{6}, r \sin \frac{\pi}{6} \right).$$

$$\therefore \frac{3r^2}{64} + \frac{r^2}{28} = 1 \Rightarrow r^2 = \frac{448}{37}$$

19. (5) Locus given by equation $\arg(z + i) = \frac{\pi}{4}$ is a ray EF as shown in the following figure.



Complex number z moves on ray EF .

Now, for any position of complex number $P(z)$ on the ray EF ,

$$|z - (-1 + i)| + |z - (2 - 3i)| = PA + PB$$

Clearly, $PA + PB$ is minimum when point P is at point C , where points A , C and B are collinear.

$$\text{So, } (PA + PB)_{\min} = AB = 5$$

$$\begin{aligned}
 \text{20. (33)} \quad (4 + 5\omega + 6\omega^2)^{n^2+2} + (6 + 5\omega^2 + 4\omega)^{n^2+2} \\
 + (5 + 6\omega + 4\omega^2)^{n^2+2} = 0
 \end{aligned}$$

$$\Rightarrow (4 + 5\omega + 6\omega^2)^{n^2+2} [1 + \omega^{n^2+2} + \omega^{2n^2+4}] = 0$$

$$\Rightarrow n = 3\lambda, n \neq 3\lambda + 1, 3\lambda + 2$$

$$\therefore n = 3, 6, 9, 12, \dots, 99$$

So, number of values of n is 33.

$$\begin{aligned}
 \text{21. (15)} \quad \sum_{k=1}^{22} \frac{1}{1 + z^{8k} + z^{16k}} &= \sum_{k=1}^{22} \frac{z^{8k} - 1}{(z^{8k} - 1)(z^{16k} + z^{8k} + 1)} \\
 &= \sum_{k=1}^{22} \frac{z^{8k} - 1}{z^{24k} - 1} \\
 &= \sum_{k=1}^{22} \frac{z^{8k} - 1}{z^k - 1} \quad (\because z^{24k} = z^{23k} z^k = 1 \cdot z^k = z^k)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=1}^{22} (1 + z^k + z^{2k} + \dots + z^{7k}) \\
 &= 22 + (0 - 1) \times 7 = 15
 \end{aligned}$$

22. (0.5) We know that

$$\begin{aligned}
 |\bar{z}_2 - z_1|^2 &= |z_2|^2 + |z_1|^2 - 2\operatorname{Re}(z_1 z_2) \\
 &= (|z_2| - |z_1|)^2 + 2(|z_1| - \operatorname{Re}(z)) \\
 \therefore |z| - \operatorname{Re}(z) &= \frac{|\bar{z}_2 - z_1|^2}{2} - \frac{(|z_2| - |z_1|)^2}{2} \\
 &\leq \frac{1}{2} - \frac{(|z_2| - |z_1|)^2}{2} \leq \frac{1}{2} \\
 \therefore (|z| - \operatorname{Re}(z))_{\max.} &= \frac{1}{2}
 \end{aligned}$$

23. (1) $z_1 + z_2 + z_3 = z_1 z_2 + z_2 z_3 + z_3 z_1 = z_1 z_2 z_3 = 1$
i.e., Sum of roots = Sum of roots taken two at a time = Product of roots = 1

So, equation will be

$$\begin{aligned}
 z^3 - z^2 + z - 1 &= 0 \\
 \Rightarrow (z - 1)(z^2 + 1) &= 0 \\
 \Rightarrow z &= 1, -i, i
 \end{aligned}$$

Area of the triangle formed by the point $A(z_1)$, $B(z_2)$

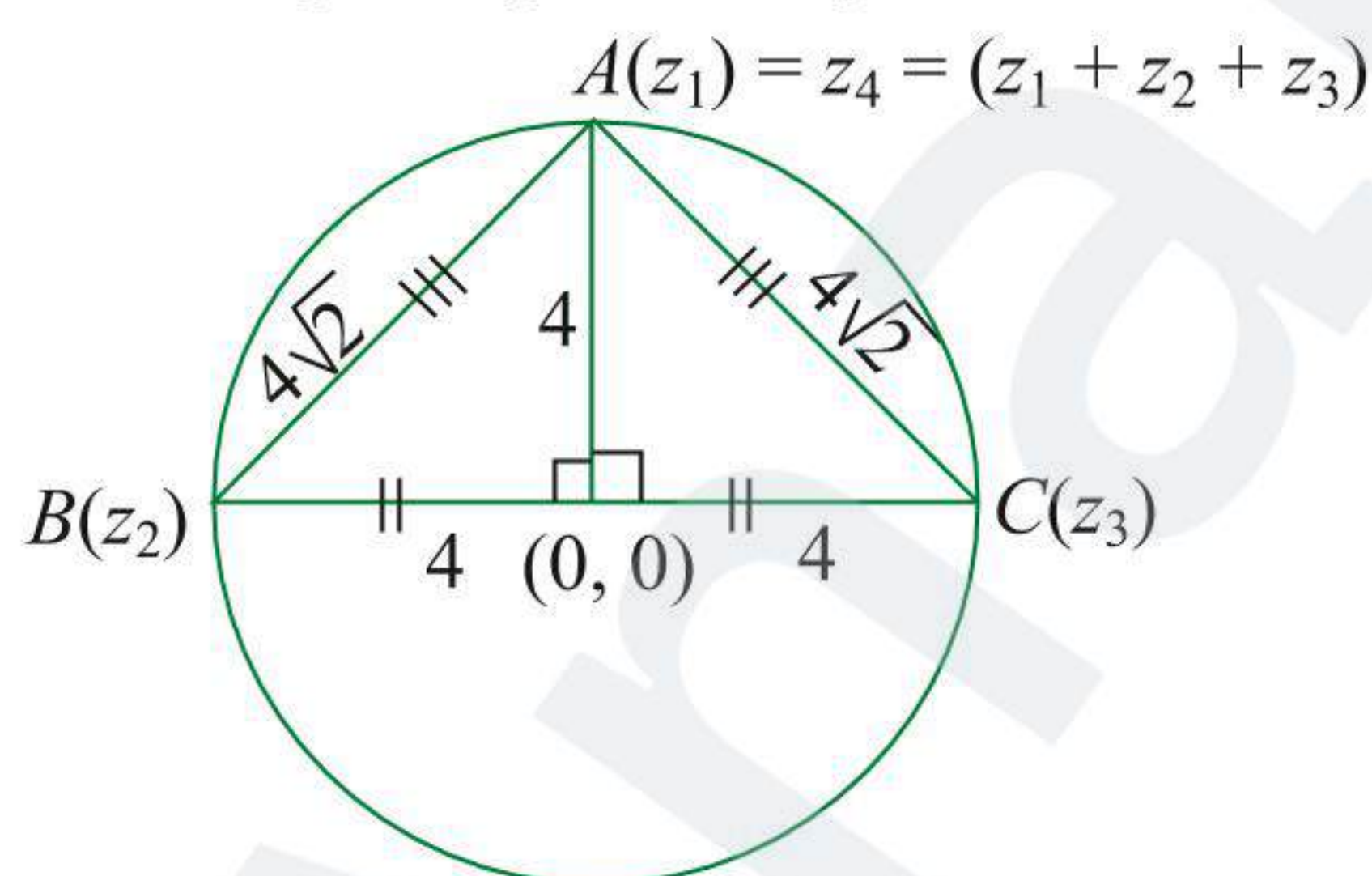
$$\text{and } C(z_3) = \frac{1}{2} \times 2 \times 1 = 1$$

24. (40) $|z_1 + \alpha^t z_2|^2 = |z_1|^2 + |z_2|^2 + \alpha^t z_2 \bar{z}_1 + (\bar{\alpha})^t \bar{z}_2 z_1$

$$\text{Now, } \sum_{t=0}^4 \alpha^t = 0 \text{ and } \sum_{t=0}^4 (\bar{\alpha})^t = 0$$

$$\begin{aligned}
 \therefore \sum_{t=0}^4 |z_1 + \alpha^t z_2|^2 &= 5(|z_1|^2 + |z_2|^2) \\
 &= 5(4 + 4) = 40
 \end{aligned}$$

25. (32) Since, $|z_1| = |z_2| = |z_3| = 4$, origin is circumcentre of triangle formed by $A(z_1)$, $B(z_2)$ and $C(z_3)$.



So, orthocentre of triangle is $z_1 + z_2 + z_3 = z_4$ (say).

Also, given that $|z_1 + z_2 + z_3| = 4$.

So, z_4 lies on the circumcircle of triangle ABC .

Given that $|z_1 - z_2| = |z_1 - z_3|$

$$\Rightarrow AB = AC$$

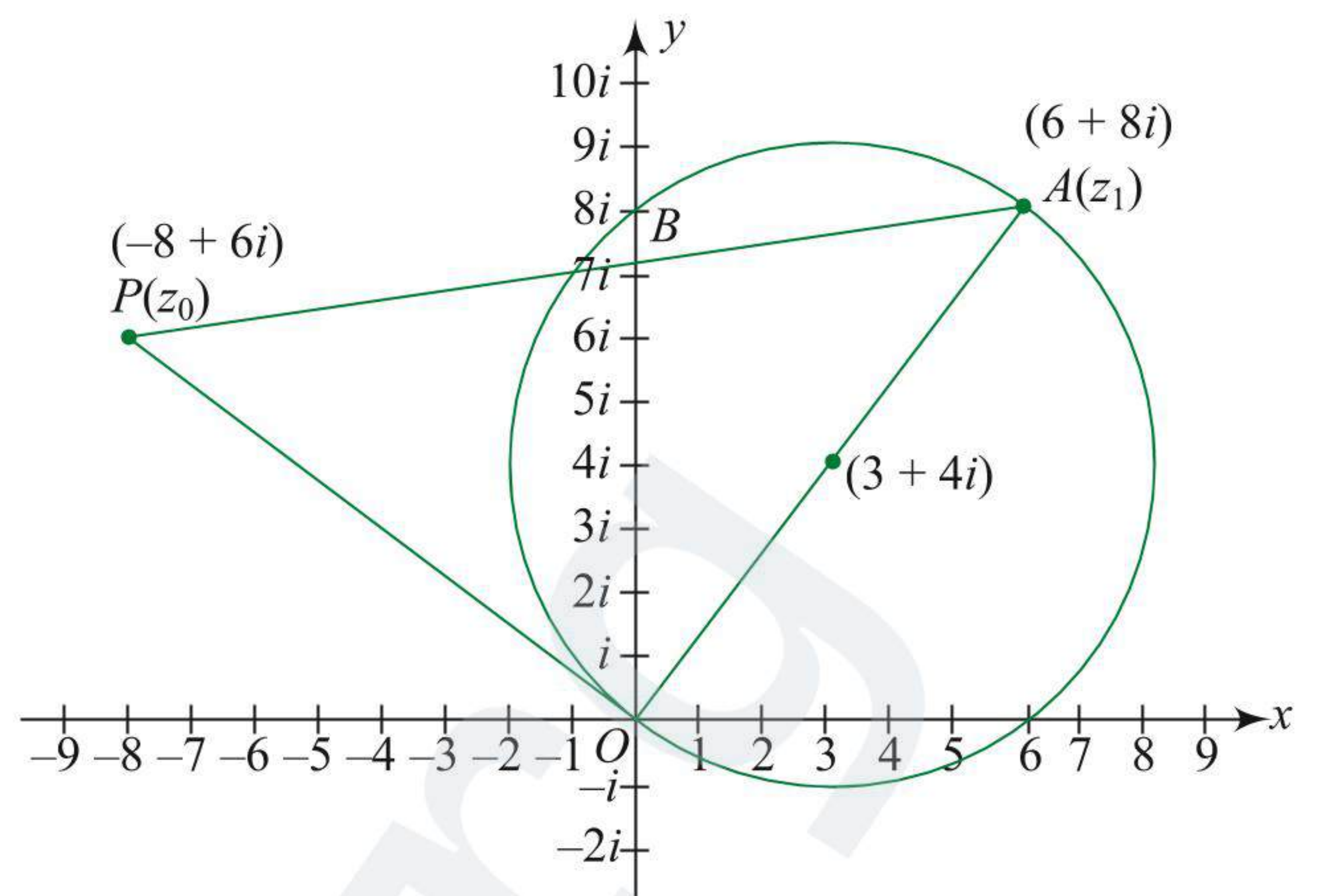
$\therefore \angle BAC = 90^\circ$ (orthocentre is at vertex A)

$$\begin{aligned}
 \text{Thus, } |z_1 + z_2| \cdot |z_1 + z_3| &= |z_4 - z_3| |z_4 - z_2| \\
 &= (AC)(AB) = (4\sqrt{2})(4\sqrt{2}) = 32
 \end{aligned}$$

26. (100)

We have $|z - (3 + 4i)| = 5$

So, z lies on the circle having center at $3 + 4i$ and radius 5.



This circle passes through the origin.

z_1 lies on circle such that $|z_1|$ is maximum.

Therefore, z_1 is at other end of the diameter through origin.

$$\therefore z_1 = 6 + 8i.$$

OA is rotated by 90° in anticlockwise direction.

$$\text{So, } z_0 = i(6 + 8i) = -8 + 6i$$

Since A, B and P are collinear,

$$|z_0 - z_1| \cdot |z_0 - z_2| = PA \cdot PB = OP^2 = 10^2 = 100$$

27. (12) $|z| = |z_2| = |z_3| = 2$

$$\text{Now, } |z_1 + z_2 + z_3|^2 \geq 0$$

$$\Rightarrow (z_1 + z_2 + z_3)(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) \geq 0$$

$$\Rightarrow \sum |z_i|^2 + \sum (z_1 \bar{z}_2 + \bar{z}_1 z_2) \geq 0$$

$$\Rightarrow 2 \sum |z_i|^2 + \sum (z_1 \bar{z}_2 + \bar{z}_1 z_2) \geq \sum |z_i|^2$$

$$\Rightarrow |z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2 \geq 12$$

28. (2) $|z_1 + 1| + |z_2 + 1| + |z_1 z_2 + 1|$

$$\geq |z_1 + 1| + |z_2 + 1 - z_1 z_2 - 1|$$

$$\geq |z_1 + 1| + |z_2| |z_1 - 1|$$

$$= |z_1 + 1| + |z_1 - 1|$$

$$\geq 2$$

29. (2) $|z_1| = 2$

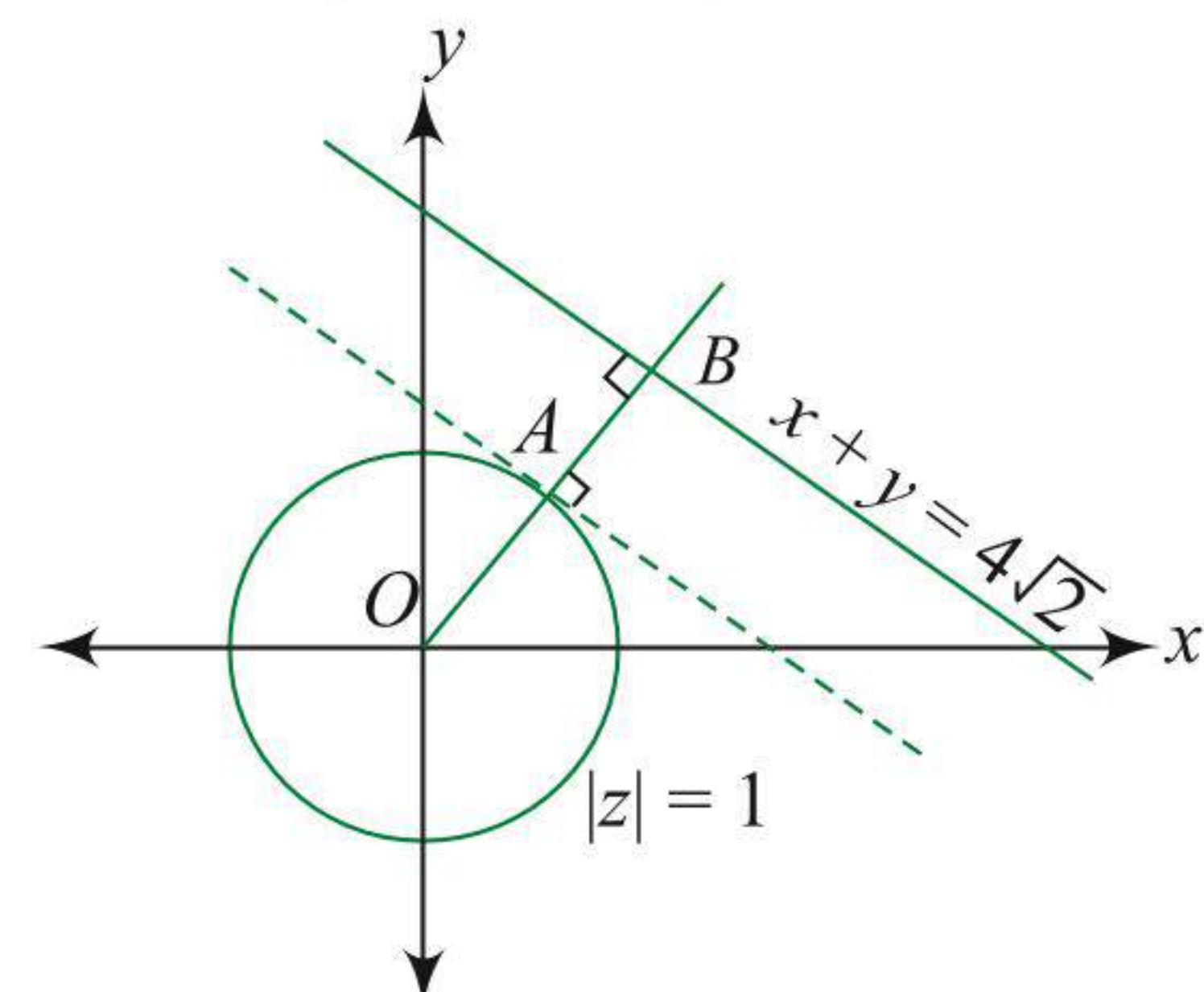
This implies that z_1 lies on the circle having center at origin and radius 2.

$$(1 - i)z_2 + (1 + i)\bar{z}_2 = 8\sqrt{2}$$

$$\therefore (1 - i)(x_2 + iy_2) + (1 + i)(x_2 - iy_2) = 8\sqrt{2}$$

$$\Rightarrow x_2 + y_2 = 4\sqrt{2}$$

So, z_2 lies on the straight line $x + y = 4\sqrt{2}$.



$$|z_1 - z_2|_{\min.} = \text{shortest distance between circle and straight line} = AB$$

$$= OB - OA = 4 - 2 = 2$$

30. (1) $1 + 2|z|^2 = |z^2 + 1|^2 + 2|z + 1|^2$

$$\Rightarrow 1 + 2z\bar{z} = (z^2 + 1)(\bar{z}^2 + 1) + 2(z + 1)(\bar{z} + 1)$$

$$\begin{aligned}
&\Rightarrow (z\bar{z})^2 + 2(z + \bar{z}) + z^2 + \bar{z}^2 + 2 = 0 \\
&\Rightarrow (z\bar{z} - 1)^2 + (z + \bar{z} + 1)^2 = 0 \\
&\Rightarrow z\bar{z} = 1 \Rightarrow \bar{z} = \frac{1}{z} \\
&\text{and } z + \bar{z} + 1 = 0 \\
&\text{or } z + \frac{1}{z} + 1 = 0 \\
&\Rightarrow z^2 + z = -1 \\
&\Rightarrow |z^2 + z| = 1
\end{aligned}$$

Archives

JEE ADVANCED

Single Correct Answer Type

1. (1) $z\bar{z}(\bar{z}^2 + z^2) = 350$

Putting $z = x + iy$, we have

$$(x^2 + y^2)(x^2 - y^2) = 175$$

$$(x^2 + y^2)(x^2 - y^2) = 5 \times 5 \times 7$$

$$x^2 + y^2 = 25$$

and $x^2 - y^2 = 7$

(as other combinations give non-integral values of x and y)

$$\therefore x = \pm 4, y = \pm 3 \quad (x, y \in \mathbb{I})$$

Hence, area is $8 \times 6 = 48$ sq. units.

2. (4) Given equation is $z^2 + z + 1 - a = 0$

Clearly this equation do not have real roots if

$$D < 0$$

$$\Rightarrow 1 - 4(1 - a) < 0$$

$$\Rightarrow 4a < 3$$

$$a < \frac{3}{4}$$

3. (3) Given circles are

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$

or $|z - z_0| = r$ (1)

and $|z - z_0| = 2r$ (2)

where $z_0 = x_0 + iy_0$

Now α and $\frac{1}{\bar{\alpha}}$ lies on circle (1) and (2), respectively. Then

$$|\alpha - z_0| = r \text{ and } \left| \frac{1}{\bar{\alpha}} - z_0 \right| = 2r$$

$$\Rightarrow |\alpha - z_0| = r \text{ and } |1 - \bar{\alpha}z_0| = 2r|\bar{\alpha}|$$

$$\Rightarrow |\alpha - z_0|^2 = r^2 \text{ and } |1 - \bar{\alpha}z_0|^2 = 4r^2|\bar{\alpha}|^2$$

Subtracting, we get

$$|1 - \bar{\alpha}z_0|^2 - |\alpha - z_0|^2 = 4r^2|\bar{\alpha}|^2 - r^2$$

$$\begin{aligned}
&\Rightarrow 1 + |\alpha z_0|^2 - \bar{\alpha}z_0 - \alpha\bar{z}_0 - (|\alpha|^2 + |z_0|^2 - \bar{\alpha}z_0 - \alpha\bar{z}_0) \\
&= 4r^2|\bar{\alpha}|^2 - r^2
\end{aligned}$$

$$\Rightarrow 1 + |\alpha|^2|z_0|^2 - |\alpha|^2 - |z_0|^2 = 4r^2|\bar{\alpha}|^2 - r^2$$

$$\Rightarrow (1 - |\alpha|^2)(1 - |z_0|^2) = 4r^2|\bar{\alpha}|^2 - r^2$$

Given $2|z_0|^2 = r^2 + 2$

$$\Rightarrow (1 - |\alpha|^2) \left(1 - \frac{r^2 + 2}{2} \right) = 4r^2|\bar{\alpha}|^2 - r^2$$

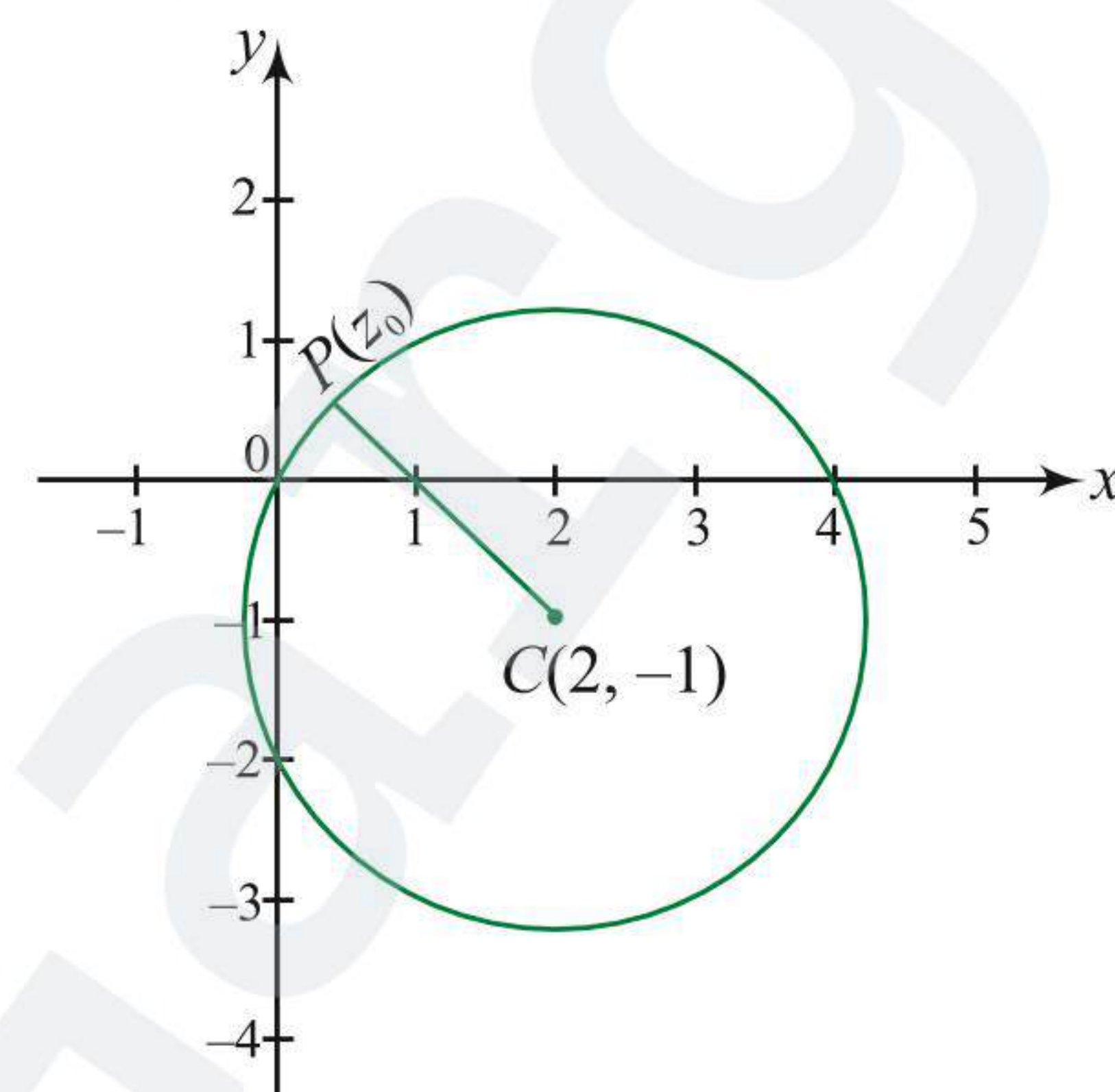
$$\Rightarrow (1 - |\alpha|^2) \left(\frac{-r^2}{2} \right) = 4r^2|\bar{\alpha}|^2 - r^2$$

$$\Rightarrow |\alpha|^2 - 1 = 8|\alpha|^2 - 2$$

$$\Rightarrow |\alpha|^2 = \frac{1}{7} \Rightarrow |\alpha| = \frac{1}{\sqrt{7}}$$

4. (3) We have $|z - (2 - i)| \geq \sqrt{5}$

So, z lies on or outside the circle having centre at $(2 - i)$ and radius, $r = \sqrt{5}$.



Now, $\frac{1}{|z_0 - 1|}$ is maximum, when $|z_0 - 1|$ is minimum, for

which $z_0 = x_0 + iy_0$ is on the circle at point P .

$$\therefore \arg \left(\frac{4 - (z_0 + \bar{z}_0)}{z_0 - \bar{z}_0 + 2i} \right) = \arg \left(\frac{4 - 2x}{2iy + 2i} \right)$$

$$= \arg \left(\frac{-i(2 - x)}{y + 2} \right)$$

$$= \arg(-i\lambda), \text{ where } \lambda > 0 \text{ as } x < 2 = -\frac{\pi}{2}$$

5. (3) $z_1 = e^{i\theta_1}$

$$\therefore |z_1| = 1$$

$$z_k = z_{k-1}e^{i\theta_k}$$

$$\therefore |z_2| = |z_1|e^{i\theta_2} = 1$$

$$\Rightarrow |z_k| = 1 \text{ for } k = 1, 2, 3, \dots, 10$$

$$S_1 = |z_2 - z_1| + |z_3 - z_2| + \dots + |z_{10} - z_9| + |z_1 - z_{10}|$$

$$= \left(\sum_{k=2}^{10} |z_k - z_{k-1}| \right) + |z_1 - z_{10}|$$

$$|z_k - z_{k-1}| = |z_{k-1}e^{i\theta_k} - z_{k-1}|$$

$$= |z_{k-1}| \cdot |e^{i\theta_k} - 1|$$

$$= \sqrt{2 - 2\cos\theta_k} = 2 \left| \sin \frac{\theta_k}{2} \right| \leq 2 \cdot \frac{\theta_k}{2} = \theta_k$$

$$\therefore \sum_{k=2}^{10} |z_k - z_{k-1}|$$

$$= \theta_2 + \theta_3 + \theta_4 + \dots + \theta_{10}$$

$$= 2\pi - \theta_1$$

Now, $z_2 = z_1 \cdot e^{i\theta_2}$

$$z_3 = z_2 \cdot e^{i\theta_3} = z_1 \cdot e^{i\theta_2} \cdot e^{i\theta_3} = z_1 e^{i(\theta_2 + \theta_3)}$$

$$\therefore z_{10} = z_1 \cdot e^{i(\theta_2 + \theta_3 + \dots + \theta_{10})}$$

$$= z_1 \cdot e^{i(2\pi - \theta_1)} = z_1 \cdot e^{i\theta_1}$$

$$\therefore |z_1 - z_{10}| = |z_1| \cdot |1 - e^{i\theta_1}| = 2 \left| \sin \frac{\theta_1}{2} \right| \leq \theta_1$$

$$\therefore S_1 \leq \theta_1 + \theta_2 + \dots + \theta_{10} = 2\pi$$

Therefore, statement P is true.

With similar reasons, statement Q is also true.

Multiple Correct Answers Type

1. (1), (3), (4)

$$\text{Given } z = (1-t)z_1 + tz_2$$

$$\Rightarrow z = \frac{(1-t)z_1 + tz_2}{(1-t) + t}$$

$\Rightarrow z$ divides the line segment joining z_1 and z_2 in ratio $(1-t) : t$ internally as $0 < t < 1$

$\Rightarrow z, z_1$, and z_2 are collinear.

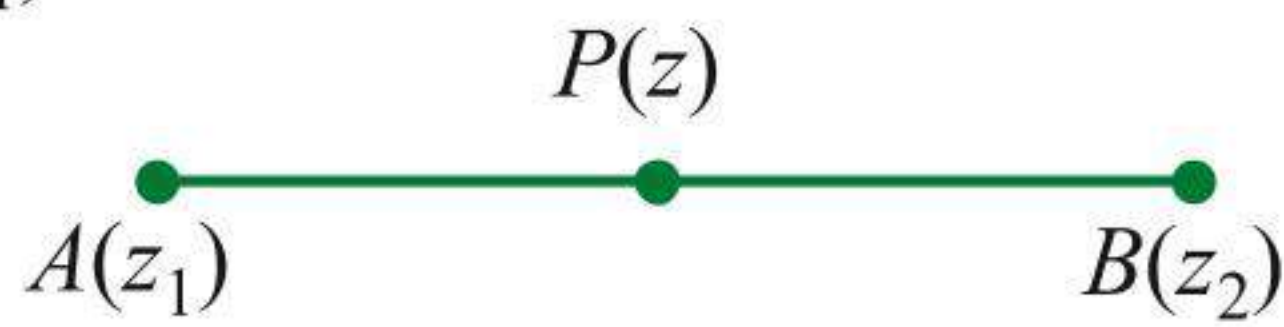
$$\Rightarrow \arg(z - z_1) = \arg(z_2 - z_1)$$

$$\Rightarrow \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$

$$\Rightarrow \left| \frac{z - z_1}{z_2 - z_1} \cdot \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \right| = 0$$

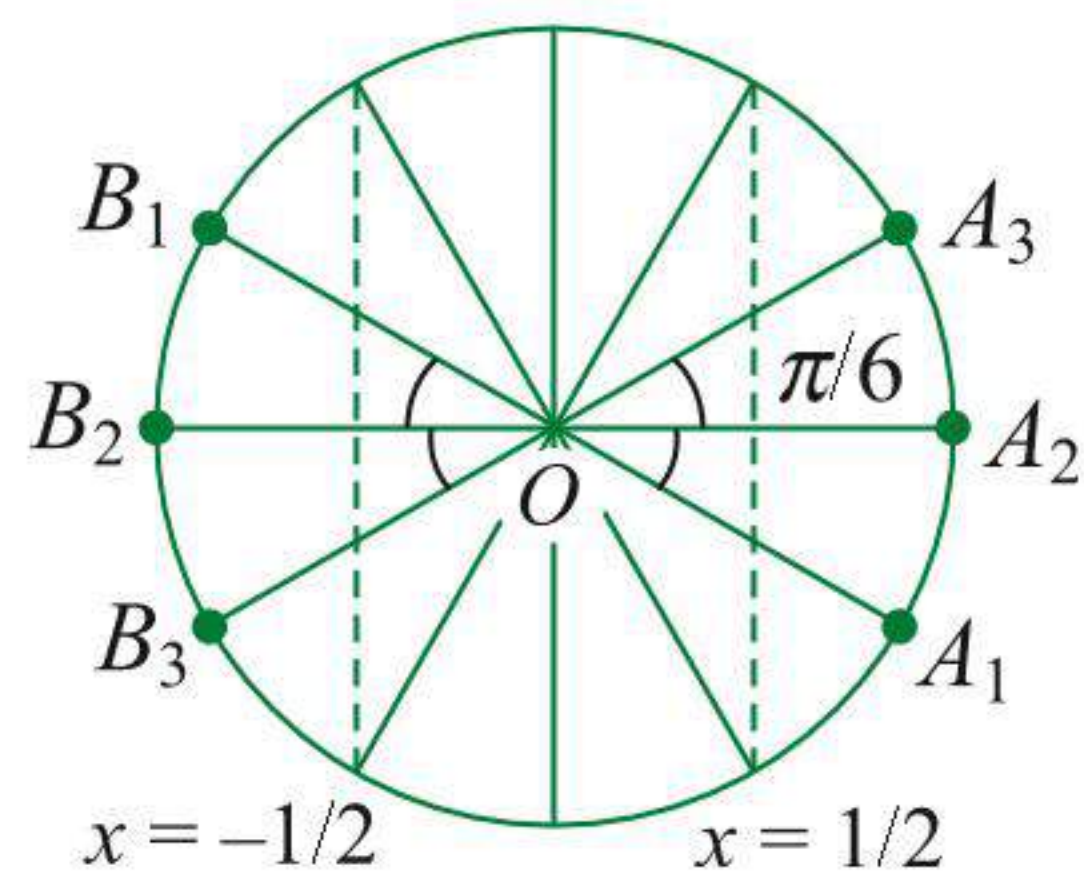
$$AP + PB = AB$$

$$\Rightarrow |z - z_1| + |z - z_2| = |z_1 - z_2|$$



2. (3), (4)

$$w = \frac{\sqrt{3} + i}{2} = e^{i\pi/6}, \text{ so } w^n = e^{i(n\pi/6)}, n = 0, 1, 2, 3, \dots, 12$$



Now, for z_1 , $\cos \frac{n\pi}{6} > \frac{1}{2}$ and for z_2 , $\cos \frac{n\pi}{6} < -\frac{1}{2}$

Possible position of z_1 are A_1, A_2, A_3 whereas of z_2 are B_1, B_2, B_3 (as shown in the figure)

So possible value of $\angle z_1 O z_2$ according to the given options is

$$\frac{2\pi}{3} \text{ or } \frac{5\pi}{6}$$

3. (1), (3), (4)

$$z = \frac{1}{a + ibt}$$

$$\Rightarrow x + iy = \frac{a - ibt}{a^2 + b^2 t^2}$$

$$\Rightarrow x = \frac{a}{a^2 + b^2 t^2}, y = \frac{-bt}{a^2 + b^2 t^2}$$

Eliminating t , we get

$$x^2 + y^2 = \frac{x}{a}$$

$$\Rightarrow \left(x - \frac{1}{2a}\right)^2 + y^2 = \left(\frac{1}{2a}\right)^2$$

$\therefore$ Option (1) is correct.

(3), (4) can be verified by putting $b = 0$ and $a = 0$ respectively.

4. (1), (4)

$$\text{We have } \operatorname{Im}\left(\frac{az+b}{z+1}\right) = y \text{ and } z = x + iy$$

$$\therefore \operatorname{Im}\left(\frac{a(x+iy)+b}{x+iy+1}\right) = y$$

$$\Rightarrow \operatorname{Im}\left(\frac{(ax+b+ia y)(x+1-iy)}{(x+1)^2 + y^2}\right) = y$$

$$\Rightarrow -y(ax+b) + ay(x+1) = y((x+1)^2 + y^2)$$

$$\Rightarrow (a-b)y = y((x+1)^2 + y^2)$$

$$\therefore y \neq 0 \text{ and } a-b = 1$$

$$\therefore (x+1)^2 + y^2 = 1$$

$$\Rightarrow x = -1 \pm \sqrt{1-y^2}$$

5. (1), (2), (4)

(1) Complex number $-1 - i$ lies in third quadrant.

$$\therefore \arg(-1 - i) = -\frac{3\pi}{4}$$

(2) We have $f: R \rightarrow (-\pi, \pi], f(t) = \arg(-1 + it)$ for all $t \in R$
Complex number $-1 + it$ lies in second quadrant if $t > 0$
and in third quadrant if $t < 0$.

$$\therefore f(t) = \arg(-1 + it)$$

$$= \begin{cases} \pi - \tan^{-1}\left|\frac{t}{-1}\right|, & t \geq 0 \\ -\left(\pi - \tan^{-1}\left|\frac{t}{-1}\right|\right), & t < 0 \end{cases}$$

$$= \begin{cases} \pi - \tan^{-1}t, & t \geq 0 \\ -\pi + \tan^{-1}t, & t < 0 \end{cases}$$

Clearly, $f(t)$ is discontinuous at $t = 0$.

$$(3) \arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$$

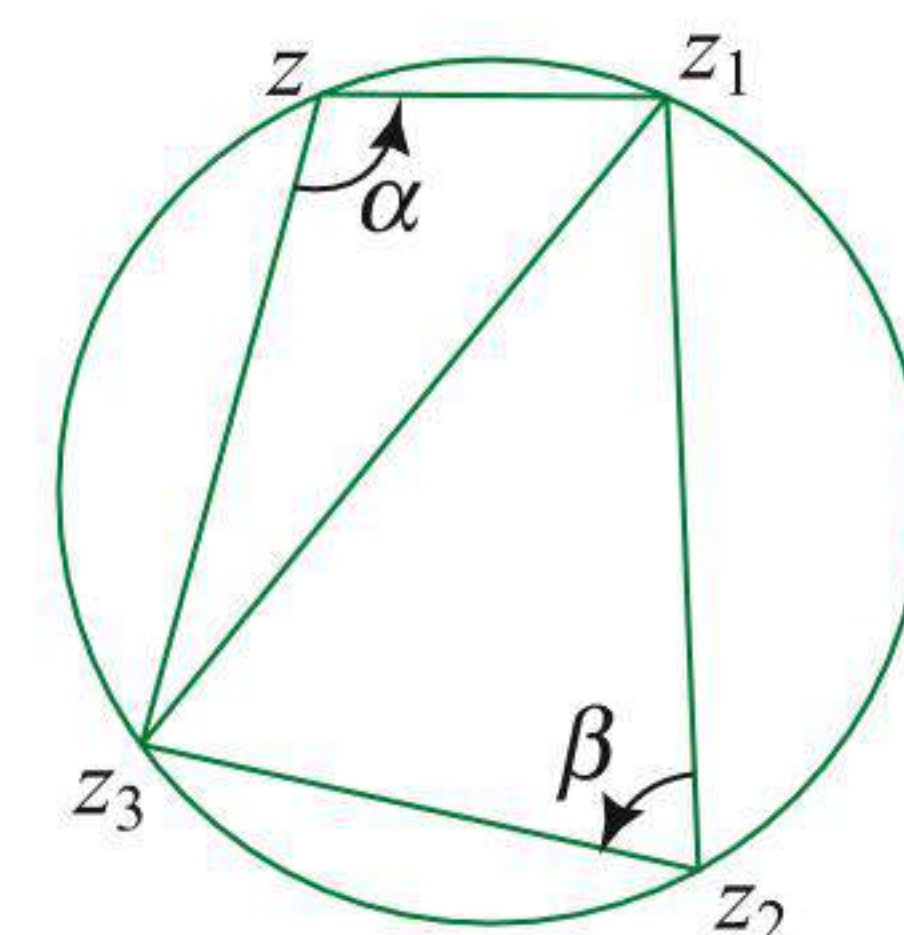
$$= \arg z_1 - \arg z_2 + 2n\pi - \arg z_1 + \arg z_2 = 2n\pi, n \in I$$

$$(4) \arg\left(\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}\right) = \pi$$

$$\Rightarrow \arg \frac{z - z_1}{z - z_3} + \arg \frac{z_2 - z_3}{z_2 - z_1} = \pi$$

Where α and β are angles subtended by line segment joining z_1 and z_3 at z and z_2 respectively.

So, we get the following figure:



Thus, locus of z is a circle.

6. (1), (3), (4)

$$\text{Given } sz + t\bar{z} + r = 0, \bar{z} = x - iy \quad \dots(i)$$

Taking conjugate, we get

$$\bar{s}\bar{z} + \bar{t}z + \bar{r} = 0 \quad \dots(ii)$$

Adding (i) and (ii), we get

$$(t + \bar{s})\bar{z} + (\bar{t} + s)z + (r + \bar{r}) = 0$$

Clearly, this is the equation of a straight line.

Now, eliminating $\bar{z}$ from (i) and (ii), we get

$$s\bar{s}z + r\bar{s} - t\bar{t}z - \bar{r}t = 0$$

$$\Rightarrow (|s|^2 - |t|^2)z = \bar{r}t - r\bar{s}$$

If $|s| \neq |t|$, then L has unique solution.

If $|s| = |t|$ and $\bar{r}t = r\bar{s}$, then L has infinite solutions.

If $|s| = |t|$ and $\bar{r}t \neq r\bar{s}$, then L has no solution.

Thus, L has either unique, infinite or no solution.

Also, line can intersect circle in maximum two points.

So, number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2.

7. (2), (3) $|z^2 + z + 1| = 1$

$$\Rightarrow \left| \left(z + \frac{1}{2} \right)^2 + \frac{3}{4} \right| = 1$$

$$\Rightarrow \left| \left(z + \frac{1}{2} \right)^2 + \frac{3}{4} \right| \leq \left| z + \frac{1}{2} \right|^2 + \frac{3}{4}$$

$$\Rightarrow 1 \leq \left| z + \frac{1}{2} \right|^2 + \frac{3}{4}$$

$$\Rightarrow \left| \left(z + \frac{1}{2} \right)^2 \right| \geq \frac{1}{4}$$

$$\Rightarrow \left| z + \frac{1}{2} \right| \geq \frac{1}{2}$$

Also, $|(z^2 + z) + 1| \geq ||z^2 + z| - 1|$

$$\Rightarrow |z^2 + z| - 1 \leq 1$$

$$\Rightarrow |z^2 + z| \leq 2$$

$$\Rightarrow ||z^2| - |z|| \leq |z^2 + z| \leq 2$$

$$\Rightarrow ||z|^2 - |z|| \leq 2$$

$$\Rightarrow -2 \leq |z|^2 - |z| \leq 2$$

$$\Rightarrow |z|^2 - |z| - 2 \leq 0$$

$$\Rightarrow (|z| - 2)(|z| + 1) \leq 0$$

$$\Rightarrow |z| \leq 2$$

Also, $|z^2 + z + 1| = 1$

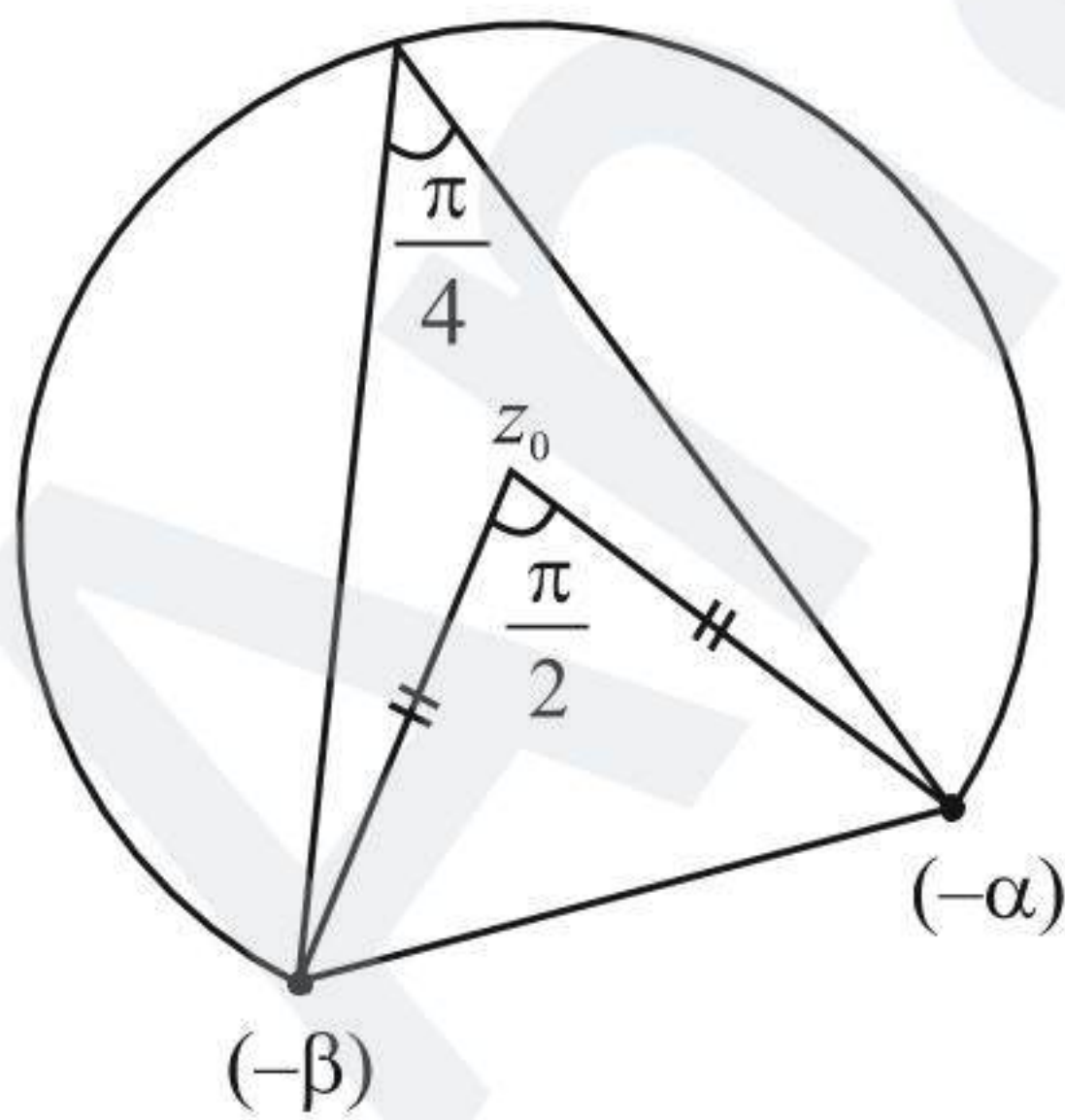
$$\Rightarrow z^2 + z + 1 = e^{i\theta}, \theta \in R$$

So, equation has infinite number of roots for different values of θ .

8. (2), (4)

Center of the circle $x^2 + y^2 + 5x - 3y + 4 = 0$ is $\left(-\frac{5}{2}, \frac{3}{2}\right)$

$$\therefore z_0 = -\frac{5}{2} + \frac{3}{2}i$$



In the figure, applying rotation formula at z_0 , we get

$$\arg \left(\frac{-\frac{5}{2} + \frac{3}{2}i + \alpha}{-\frac{5}{2} + \frac{3}{2}i + \beta} \right) = \frac{\pi}{2} = \arg(i)$$

$$\therefore \frac{-\frac{5}{2} + \frac{3}{2}i + \alpha}{-\frac{5}{2} + \frac{3}{2}i + \beta} = i$$

$$\Rightarrow -\frac{5}{2} + \frac{3}{2}i + \alpha = -\frac{5}{2}i - \frac{3}{2} + i\beta$$

$$\Rightarrow \alpha - i\beta = 1 - 4i$$

$$\Rightarrow \alpha = 1, \beta = 4$$

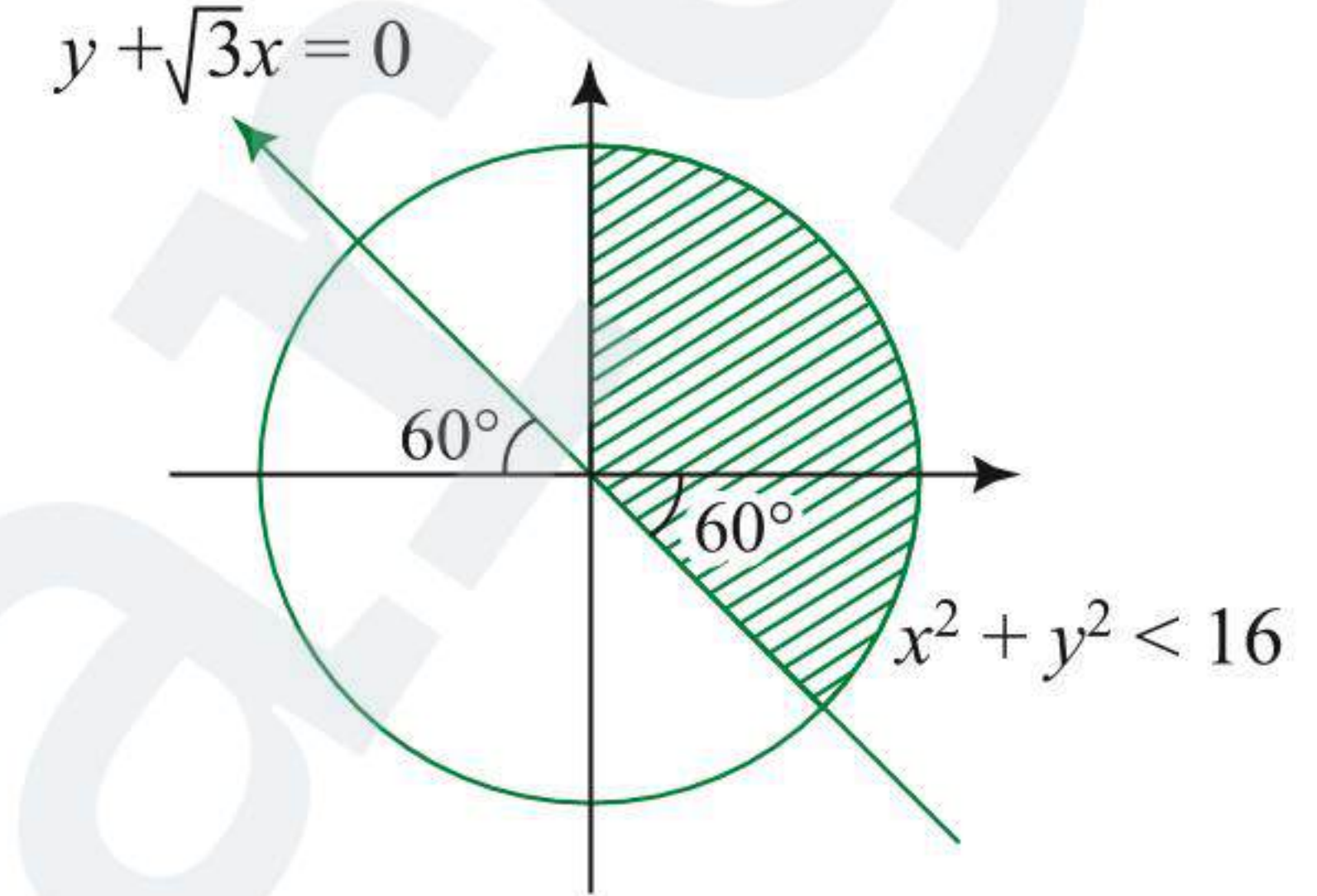
$$\Rightarrow \alpha\beta = 4$$

Linked Comprehension Type

1. (2) $S_1 : |z| < 4$, z lies inside the circle or radius 4

$S_2 : \sqrt{3}x + y > 0$, z lies above the line $\sqrt{3}x + y = 0$

$S_3 : \operatorname{Re}(z) > 0$, z lies to the right of imaginary axis.



Area of region $S_1 \cap S_2 \cap S_3 = \text{Shaded area}$

$$= \frac{\pi \times 4^2}{4} + \frac{4^2 \times \pi}{6} = 4^2 \pi \left\{ \frac{1}{4} + \frac{1}{6} \right\} = \frac{20\pi}{3}$$

2. (3) Distance of $(1, -3)$ from $y + \sqrt{3}x = 0$ is

$$\left| \frac{\sqrt{3} \times 1 - 3}{2} \right| = \frac{3 - \sqrt{3}}{2}$$

$$\Rightarrow \min_{z \in S} |1 - 3i - z| = \frac{3 - \sqrt{3}}{2}$$

Matrix Match Type

1. a $\rightarrow$ q

$$\left| \frac{z}{|z|} - i \right| = \left| \frac{z}{|z|} + i \right|, z \neq 0$$

$\frac{z}{|z|}$ is unimodular complex number and lies on perpendicular

bisector of i and $-i$

$$\Rightarrow \frac{z}{|z|} = \pm 1 \Rightarrow z = \pm 1|z|$$

$$\Rightarrow z \text{ is real number} \Rightarrow \operatorname{Im}(z) = 0.$$

b $\rightarrow$ p

$$|z + 4| + |z - 4| = 10$$

z lies on an ellipse whose focus are $(4, 0)$ and $(-4, 0)$ and length of major axis is 10

$$\Rightarrow 2ae = 8 \text{ and } 2a = 10 \Rightarrow e = 4/5$$

$$|\operatorname{Re}(z)| \leq 5.$$

c $\rightarrow$ p, t

$$|\omega| = 2 \Rightarrow \omega = 2(\cos \theta + i \sin \theta)$$

$$\Rightarrow z = x + iy = 2(\cos \theta + i \sin \theta) - \frac{1}{2}(\cos \theta - i \sin \theta)$$

$$= \frac{3}{2}\cos \theta + i\frac{5}{2}\sin \theta \Rightarrow \frac{x^2}{(3/2)^2} + \frac{y^2}{(5/2)^2} = 1$$

$$\Rightarrow e^2 = 1 - \frac{9/4}{25/4} = 1 - \frac{9}{25} = \frac{16}{25} \Rightarrow e = \frac{4}{5}$$

d $\rightarrow$ q, t

$$|\omega| = 1 \Rightarrow x + iy = \cos \theta + i \sin \theta + \cos \theta - i \sin \theta$$

$$x + iy = 2\cos \theta$$

$$|\operatorname{Re}(z)| \leq 1, \operatorname{Im}(z) = 0.$$

2. (1) a. z_k is 10^{th} root of unity

So, $\bar{z}_k$ will also be 10^{th} root of unity.

Take z_j as $\bar{z}_k$.

b. $z_1 \neq 0$ take $z = \frac{z_k}{z_1}$, we can always find z .

c. $z^{10} - 1 = (z - 1)(z - z_1) \dots (z - z_9)$
 $\Rightarrow (z - z_1)(z - z_2) \dots (z - z_9)$
 $= 1 + z + z^2 + \dots + z^9 \quad \forall z \in \text{complex number}$

Put $z = 1$

$$\Rightarrow (1 - z_1)(1 - z_2) \dots (1 - z_9) = 10$$

$$\Rightarrow \frac{|1 - z_1||1 - z_2| \dots |1 - z_9|}{10} = 1$$

d. $1 + z_1 + z_2 + \dots + z_9 = 0$

$$\Rightarrow \text{Re}(1) + \text{Re}(z_1) + \dots + \text{Re}(z_9) = 0$$

$$\Rightarrow \text{Re}(z_1) + \text{Re}(z_2) + \dots + \text{Re}(z_9) = -1.$$

3. c $\rightarrow$ p, q, s, t.

$$\text{Let } a = 3 - 3\omega + 2\omega^2$$

$$\Rightarrow a\omega = 3\omega - 3\omega^2 + 2$$

$$\Rightarrow a\omega^2 = 3\omega^2 - 3 + 2\omega$$

$$\text{Now given } a^{4n+3} (1 + \omega^{4n+3} + (\omega^2)^{4n+3}) = 0$$

Thus, n should not be a multiple of 3.

Note: Solutions of the remaining parts are given in their respective chapters.

Numerical Value Type

1. (1) $\omega = e^{i2\pi/3}$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

Applying $(C_1 \rightarrow C_1 + C_2 + C_3)$

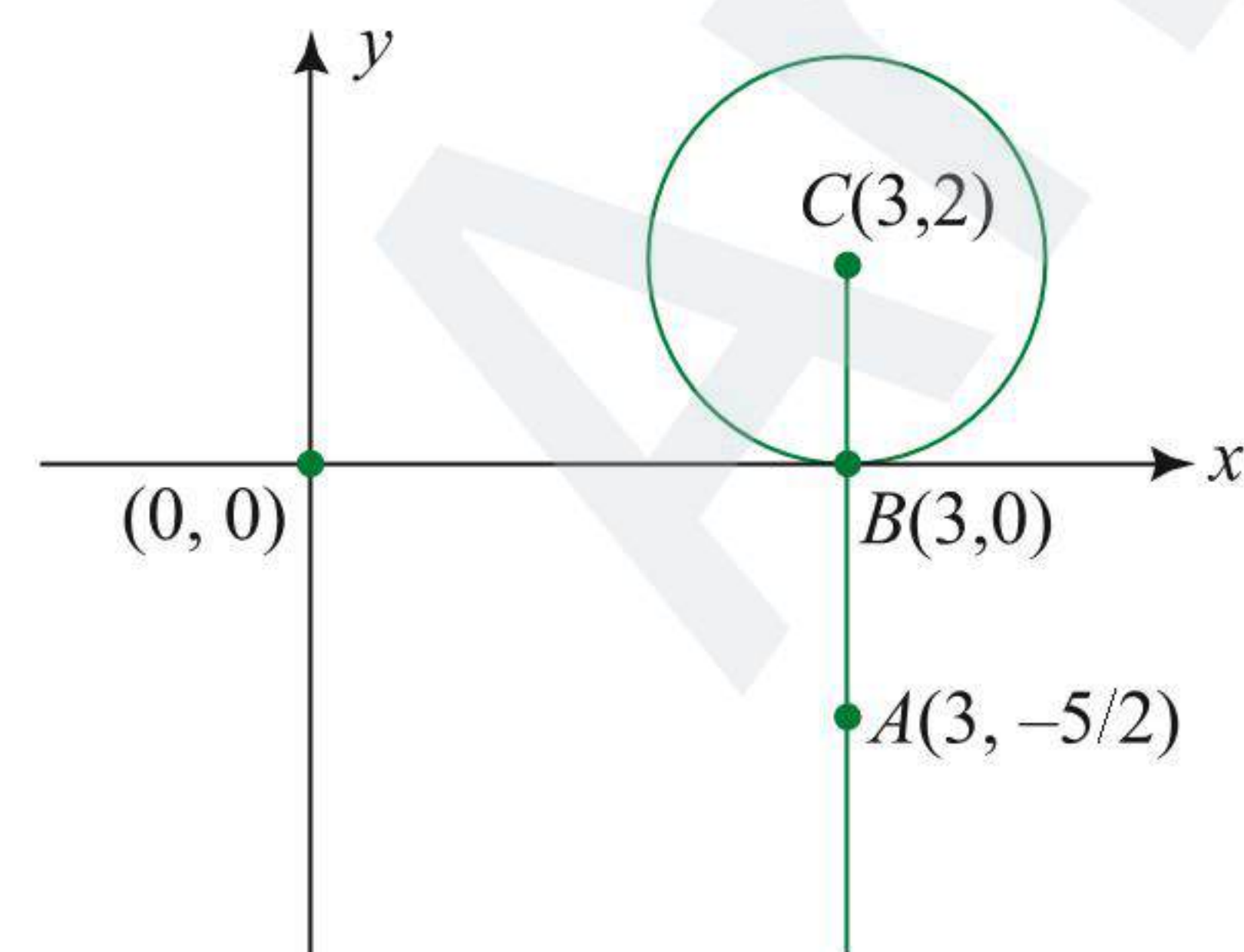
$$\Rightarrow z \begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$$\Rightarrow z^3 = 0$$

$z = 0$ is only solution.

2. (5) $|z - 3 - 2i| \leq 2$

$\Rightarrow z$ lies on or inside the circle radius 2 and center $(3, 2)$



$$|2z - 6 + 5i|_{\min.}$$

$$= 2|z - 3 + (5/2)i|_{\min.}$$

= 2(minimum distance of any point on the circle to the point $(3, -5/2)$)

$$= 2(5/2) = 5$$

3. (4) $\alpha_k = \cos \frac{k\pi}{7} + i \sin \frac{k\pi}{7} = e^{\frac{k\pi}{7}i}$

$$\begin{aligned} & \sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k| \\ \therefore & \frac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^3 |\alpha_{4k-1} - \alpha_{4k-2}|} \\ &= \frac{\sum_{k=1}^{12} \left| e^{\frac{(k+1)\pi}{7}i} - e^{\frac{k\pi}{7}i} \right|}{\sum_{k=1}^3 \left| e^{\frac{(4k-1)\pi}{7}i} - e^{\frac{(4k-2)\pi}{7}i} \right|} = \frac{\sum_{k=1}^{12} \left| e^{\frac{k\pi}{7}i} \right| \left| e^{\frac{\pi}{7}i} - 1 \right|}{\sum_{k=1}^3 \left| e^{\frac{(4k-2)\pi}{7}i} \right| \left| e^{\frac{\pi}{7}i} - 1 \right|} \\ &= \frac{\sum_{k=1}^{12} 1}{\sum_{k=1}^3 1} \left(\because \left| e^{\frac{k\pi}{7}i} \right| = \left| e^{\frac{(4k-2)\pi}{7}i} \right| = 1 \right) \\ &= \frac{12}{3} = 4 \end{aligned}$$

4. (3)

$$\begin{aligned} E &= |a + b\omega + c\omega^2|^2 \\ &= (a + b\omega + c\omega^2)(a + b\bar{\omega} + c\bar{\omega}^2) \\ &= a^2 + b^2 + c^2 - ab - bc - ca \\ &= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] \end{aligned}$$

Let $a > b > c$

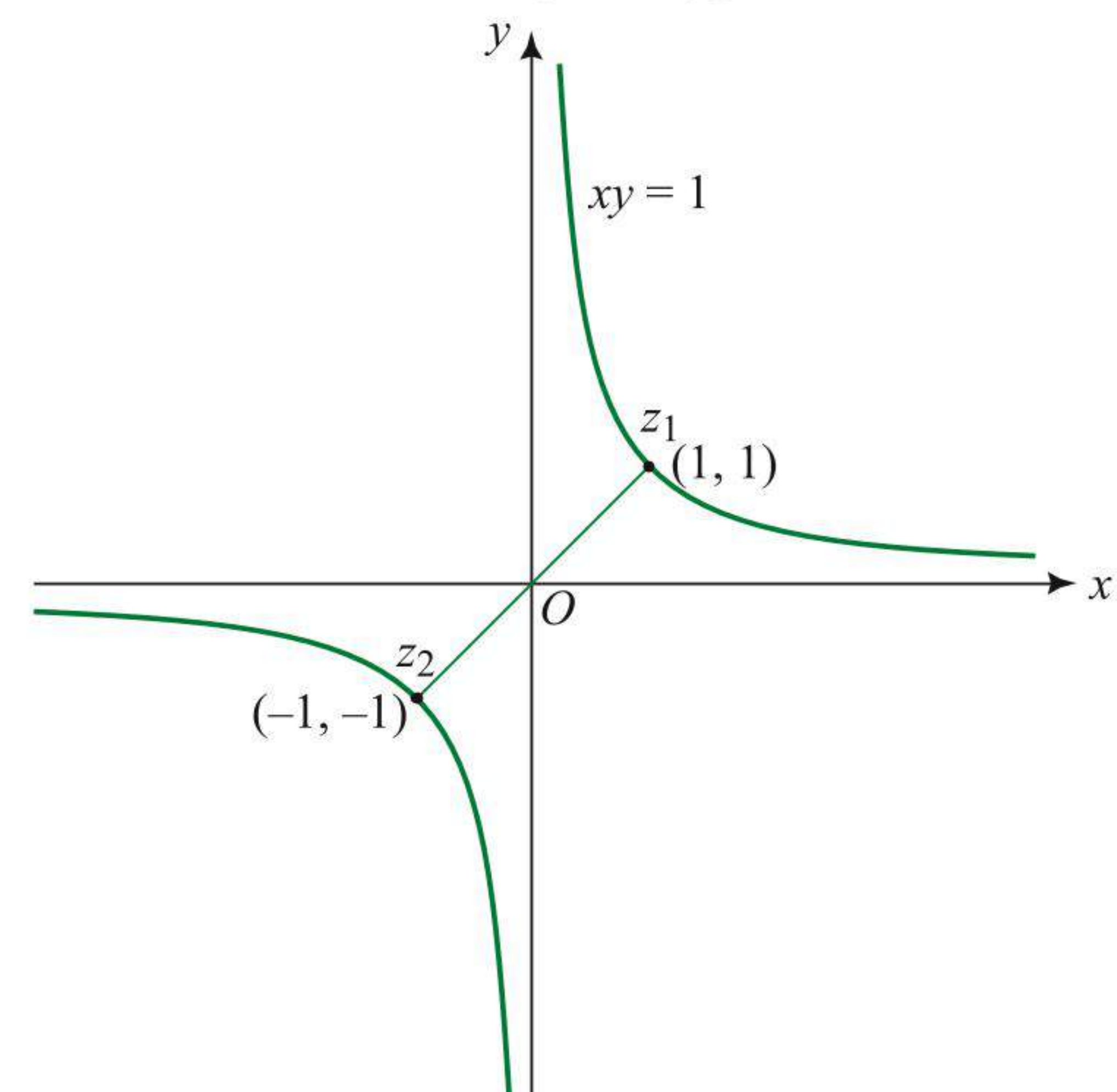
$$\Rightarrow |a-b| \geq 1, |b-c| \geq 1, |a-c| \geq 2$$

$$\text{So, } E \geq \frac{1}{2}(1 + 1 + 4) \geq 3$$

5. (8) Let $z = x + iy$

$$\begin{aligned} & z^4 - |z|^4 = 4iz^2 \\ \Rightarrow & z^4 - z^2\bar{z}^2 = 4iz^2 \\ \Rightarrow & z = 0 \text{ or } z^2 - \bar{z}^2 = 4i \\ \Rightarrow & z = 0 \text{ or } 4xyi = 4i \\ \Rightarrow & z = 0 \text{ or } xy = 1 \\ \Rightarrow & 4ixy = 4i \\ \Rightarrow & xy = 1 \end{aligned}$$

So, locus of z is rectangular hyperbola.



$$\therefore |z_1 - z_2|_{\min}^2 = 8$$